

PULSE PROPAGATION IN TIME DEPENDENT RANDOMLY LAYERED MEDIA

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Abstract. We study cumulative scattering effects on wave front propagation in time dependent randomly layered media. It is well known that the wave front has a deterministic characterization in time independent media, aside from a small random shift in the travel time. That is, the pulse shape is predictable, but faded and smeared as described mathematically by a convolution kernel determined by the autocorrelation of the random fluctuations of the wave speed. The main result of this paper is the extension of the pulse stabilization results to time dependent randomly layered media. When the media change slowly, on time scales that are longer than the pulse width and the time it takes the waves to traverse a correlation length, the pulse is not affected by the time fluctuations. In rapidly changing media, where these time scales are similar, both the pulse shape and the random component of the arrival time are affected by the statistics of the time fluctuations of the wave speed. We obtain an integral equation for the wave front, that is more complicated than in time independent media, and cannot be solved analytically, in general. We also give examples of media where the equation simplifies, and the wave front can be analyzed explicitly. We illustrate with these examples how the time fluctuations feed energy into the pulse.

Key words. randomly layered media, time dependent, pulse stabilization.

1. Introduction. We study wave front propagation in time dependent, randomly layered media. The initial condition corresponds to a plane wave normally incident to the layers, so that the mathematical model for wave propagation is the one dimensional wave equation. Extensions to three dimensional wave fields generated by spatially localized sources are not considered here, but may be done with plane wave decompositions that lead to a family of one dimensional problems, as explained in [8, Chapter14]. The medium fluctuations are around a reference speed c_o and are modeled with a random process $\nu(t, z)$ that is statistically stationary in time t and homogeneous in range z , that is, the propagation direction. The problem is to describe how a pulse impinging on the layered medium changes due to cumulative scattering effects. The answer depends on the strength of the wave speed fluctuations and the relation between the fundamental length and time scales in the problem: The distance of propagation L , the typical wavelength λ_o , the correlation length ℓ of the layers, the localization length L_{loc} , and the correlation time T_ν that quantifies the lifespan of a given spatial realization of ν . Localization means that the waves cannot penetrate too deep in the random medium, and the incident energy is scattered back. In layered media a pulse is being transformed as it propagates a distance that is on the scale of the localization length as described by the pulse stabilization theory in [1, 4, 5, 18] and in this paper.

There are two high frequency regimes ($\lambda_o \ll L$) that have been studied extensively in the context of wave propagation in time independent randomly layered media (i.e., $\nu = \nu(z)$) because they capture wave scattering effects in a canonical way [1, 4, 8]. They assume distances of propagation comparable to the localization length ($L \sim L_{loc}$) so that there is significant energy transfer between the forward going and backscattered waves, and the regimes differ by the strength of the fluctuations and the relation between the wavelength and the correlation length [8, Section 5.1.4]. In the *weakly heterogeneous* regime the wave speed fluctuations are small and $\ell \sim \lambda_o$, whereas in the *strongly heterogeneous* regime the fluctuations are of order one and $\ell \ll \lambda_o$.

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Interestingly, the wave front has a similar, nearly deterministic characterization in both regimes, as shown in [1, 4, 5, 18]. The travel time is only slightly affected by the fluctuations, via a small random shift defined by a standard Brownian motion, and the pulse shape is a faded and smeared version of the initial one, described by a deterministic convolution kernel determined by the autocorrelation of $\nu(z)$. This is called *pulse stabilization* and was first noted by O’Doherty and Anstey in a geophysical context [15]. A generalization of the pulse stabilization phenomenon persists in media with long range correlations and in multi-scale media [10, 17].

We consider time dependent randomly layered media. The scaling regime is the weakly heterogeneous regime. The waves interact strongly with the medium in this regime, because the wavelength is on the scale of the medium fluctuations, and the behavior of the pulse depends on the full autocorrelation function of the fluctuations. In the strongly heterogeneous regime the waves do not see the layers in detail, because $\lambda_o \gg \ell$, and the fluctuations take the “effective” form of white noise, independent of the detailed structure of the random process ν . The main result of the paper is the characterization of the wave front in time dependent media. We show that as long as the correlation time T_ν is long in comparison with the time $T_\ell = \ell/c_o$ of travel over a correlation length, the time changes in the medium have no effect on the pulse. The pulse perceives the medium as if it were time independent. In rapidly changing media, where $T_\nu \sim T_\ell$, the wave front has a more complicated behavior, since the medium is being transformed during the passage time of the pulse. We derive an integral equation for the wave front. It has a deterministic kernel shape, however with a random dilation, reflecting the random medium modulation during the pulse passage time. This equation is no longer solvable analytically, in general. Nevertheless, there are examples of rapidly changing media where the equation simplifies and the wave front has an explicit and deterministic expression. We illustrate with such examples how the temporal fluctuations feed energy into the wave front, thus enhancing the pulse. That is to say, there is a trade-off between pulse fading due to the spatial fluctuations and pulse enhancement due to the rapid time changes in the medium.

Wave propagation in time dependent randomly layered media has also been considered in [9], for correlation times T_ν that are small with respect to the mean scattering time*, but larger than T_ℓ . The formal stochastic iterative approach in [9] gives closed equations for the statistical moments of the forward and backward going wave fields. However, it relies on the assumption that the wave fields are statistically independent from the medium fluctuations in certain range intervals, which is difficult to justify rigorously.

The paper is organized as follows: We begin in section 2 with the mathematical formulation of the problem and the asymptotic scaling regime. The statement of the pulse stabilization result is in section 3, and its proof is in section 4. We end with a summary in section 5.

2. Formulation of the problem. We consider the same one dimensional acoustic wave equation in time changing media as in [9]

$$\begin{aligned} \rho \frac{\partial u(t, z)}{\partial t} + \frac{\partial p(t, z)}{\partial z} &= F(t, z), \\ \frac{1}{K(t, z)} \frac{\partial p(t, z)}{\partial t} + \frac{\partial u(t, z)}{\partial z} &= 0. \end{aligned} \tag{2.1}$$

*The mean scattering time is defined in [9, Equation (1.10)]. It is an order one time scale in our regime.

It can be derived from the linearization of the mass and linear momentum conservation equations in fluid dynamics, with u the displacement of the particles in the medium and p the time integral of the acoustic pressure. The bulk modulus K varies with time t and range z . The density ρ is assumed constant for simplicity, but the results can be extended to variable ρ . We model the variations of K , and thus of the wave speed $c = \sqrt{K/\rho}$, with the random process $\nu(t, z)$

$$\frac{1}{c(t, z)} = \begin{cases} [1 + \nu(t, z)]/c_o, & \text{for } z > 0, \\ 1/c_o & \text{otherwise,} \end{cases} \quad (2.2)$$

and take for simplicity a constant reference wave speed c_o . At any time instant $t \geq 0$ the fluctuations are confined to the half line $z > 0$.

The medium is quiescent for $t < 0$

$$p(t, z) = 0, \quad u(t, z) = 0, \quad t < 0, \quad (2.3)$$

and the waves are generated by a source located at $z = 0$ emitting a pulse $\mathcal{F}(t)$,

$$F(t, z) = \zeta_o^{1/2} \mathcal{F}(t) \delta(z/L), \quad (2.4)$$

for L the range scale of propagation. Here $\zeta_o = \rho c_o$ is the reference acoustic impedance, and we normalize (2.4) with it and L to simplify the formulas below. The source generates a left (backward) going wave that never interacts with the layered medium, and a right (forward) going wave which penetrates the medium. The problem is to describe the wave front, that is the right going wave observed near the expected travel time $\tau = z/c_o$ for $z > 0$, on a time scale comparable to the width of the pulse $\mathcal{F}(t)$.

2.1. Random model of the fluctuations. The random process $\nu(t, z)$ is a mathematical model of small scale fluctuations of the wave speed. Small scale means that $c(t, z)$ varies rapidly in z on the range scale L of propagation of the waves, and it changes over time intervals that are smaller than the travel time scale $T = L/c_o$. The fluctuations are not known in detail, as is typical in applications, and the random process $\nu(t, z)$ models their uncertainty.

We take ν to be a mean zero stationary random field with standard deviation σ

$$\mathbb{E} \{ \nu(t, z) \} = 0, \quad \sigma = \sqrt{\mathbb{E} \{ \nu^2(t, z) \}}, \quad (2.5)$$

and assume that it is bounded, so that the right hand side in (2.2) remains positive. Moreover, ν is twice continuously differentiable with bounded first and second order derivatives, and with rapid decay of correlations[†]. We scale its range dependence by the correlation length ℓ , and its time dependence by the correlation time T_ν . These are related to the covariance of ν as

$$\ell = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} dz' \mathbb{E} \{ \nu(t, z) \nu(t, z + z') \}, \quad T_\nu = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} dt' \mathbb{E} \{ \nu(t, z) \nu(t + t', z) \}. \quad (2.6)$$

2.2. Scaling. We model the pulse $\mathcal{F}(t)$ using a real valued function $f(s)$ of dimensionless argument

$$\mathcal{F}(t) = f(t/T_\mathcal{F}), \quad (2.7)$$

[†]See equation (3.10) for a more precise statement of the mixing assumption on the fluctuations.

and denote by $T_{\mathcal{F}}$ the time scale that quantifies the pulse duration. We assume that f is continuously differentiable and compactly supported in the interval $[0, S]$, with Fourier transform $\hat{f}(w)$ that peaks at $|w| = 2\pi$ and is supported away from the origin. Since f is real valued, $\hat{f}(-w)$ is defined by the complex conjugate of $\hat{f}(w)$. The Fourier transform of the pulse

$$\hat{\mathcal{F}}(\omega) = \int_{-\infty}^{\infty} dt f\left(\frac{t}{T_{\mathcal{F}}}\right) e^{i\omega t} = T_{\mathcal{F}} \hat{f}(\omega T_{\mathcal{F}}) \quad (2.8)$$

has the central frequency $\omega_o = 2\pi/T_{\mathcal{F}}$, which gives the central wavelength $\lambda_o = c_o T_{\mathcal{F}}$.

Let us rewrite the fluctuations in scaled form

$$\nu(t, z) = \sigma \mu\left(\frac{t}{T_{\nu}}, \frac{z}{\ell}\right), \quad (2.9)$$

using the function μ of dimensionless arguments. Its autocorrelation is given by

$$\Phi(t', z') = \mathbb{E} \{ \mu(t'' + t', z'' + z') \mu(t'', z'') \}, \quad (2.10)$$

for dimensionless $t', z' \in \mathbb{R}$. It is integrable and normalized by

$$\Phi(0, 0) = 1, \quad \int_{-\infty}^{\infty} dt' \Phi(t', 0) = 1, \quad \int_{-\infty}^{\infty} dz' \Phi(0, z') = 1, \quad (2.11)$$

to be consistent with definitions (2.5)-(2.6).

The reference range scale is \bar{L} , the typical distance of propagation through the medium, and the time is scaled by $\bar{T} = \bar{L}/c_o$. We assume that we observe the waves for a maximum time $T \sim \bar{T}$, where “ \sim ” denotes equal up to a dimensionless order one constant, and use causality to truncate mathematically the fluctuations for $z > L$. This is because until time T the waves are not affected by the medium beyond the range $L \approx c_o T$.

In the scaled coordinates $t' = t/\bar{T}$ and $z' = z/\bar{L}$, the pulse becomes

$$\mathcal{F}(t) = f\left(\frac{t'}{T_{\mathcal{F}}/\bar{T}}\right), \quad (2.12)$$

and the model of the wave speed is

$$\frac{c_o}{c(t, z)} = \begin{cases} \left[1 + \sigma \mu\left(\frac{t'}{T_{\nu}/\bar{T}}, \frac{z'}{\ell/\bar{L}}\right)\right] / c'_o, & \text{for } z' > 0, \quad z' \leq L' = L/\bar{L} \sim 1, \\ 1/c'_o & \text{otherwise.} \end{cases} \quad (2.13)$$

In our case $c'_o = 1$ because the reference speed is constant, but in general it would be a dimensionless function of z' . We keep c'_o in the equations to distinguish between the range and time variables, even though $c'_o = 1$.

2.3. The asymptotic regime. We assume from now on that the time, range and speed are scaled and drop the primes. Our analysis is in an asymptotic regime modeled with the small parameter ε defined as

$$\varepsilon^2 = T_{\mathcal{F}}/\bar{T} = \lambda_o/\bar{L} \ll 1. \quad (2.14)$$

It corresponds to a long distance of propagation compared to the wavelength. The fluctuations of the wave speed are weak, with standard deviation[‡] $\sigma = \varepsilon$, but they

[‡]The standard deviation does not have to be exactly ε , but of order ε , that is $\sigma = \sigma_o \varepsilon$, for some $\sigma_o = O(1)$. For simplicity we set $\sigma_o = 1$. A similar simplification is made in equation (2.14).

have a significant net scattering effect when the correlation length ℓ is similar to λ_o , or the correlation time T_ν is similar to the pulse width $T_{\mathcal{F}}$. For example, it is shown in [8, Chapter 14] that in time independent media with $\ell = \lambda_o$, the localization length [8, Section 5.1.4] L_{loc} is similar to our reference order one length scale \bar{L} ,

$$\frac{L_{loc}}{\bar{L}} \sim \frac{1}{\sigma^2 \bar{L} (2\pi/\lambda_o)^2 \ell} = \frac{1}{4\pi^2}. \quad (2.15)$$

This means that the net scattering is so strong that the waves cannot penetrate much deeper than the range \bar{L} in the medium, and are reflected back toward the source.

We model the range and temporal scales of the fluctuations by

$$\varepsilon^\beta = \ell/\bar{L}, \quad \varepsilon^\alpha = T_\nu/\bar{T}, \quad (2.16)$$

using the dimensionless parameters α and β in the interval $[0, 2]$. When $\alpha < 2$ we say that the medium is slowly changing, because it varies little during the duration of the pulse. In rapidly changing media $\alpha = 2$. Similarly, $\beta < 2$ models a medium that varies slowly in z .

In our scaling the pulse becomes $f(t/\varepsilon^2)$ and the model of the wave speed in the random medium is

$$c_\varepsilon(t, z) = c_o \left[1 + \varepsilon \mu \left(\frac{t}{\varepsilon^\alpha}, \frac{z}{\varepsilon^\beta} \right) \right]^{-1}, \quad (2.17)$$

where we use the index ε to remind us of the dependence on the asymptotic parameter $\varepsilon \ll 1$. The acoustic impedance is given by the product of the wave speed and the constant medium density, so it takes the form

$$\zeta_\varepsilon(t, z) = \zeta_o \left[1 + \varepsilon \mu \left(\frac{t}{\varepsilon^\alpha}, \frac{z}{\varepsilon^\beta} \right) \right]^{-1}. \quad (2.18)$$

3. Statement of results. The goal of the paper is to characterize the wave front in the asymptotic limit $\varepsilon \rightarrow 0$. To define it we introduce the right and left going waves $A_\varepsilon(t, z)$ and $B_\varepsilon(t, z)$, moving with the local speed $c_\varepsilon(t, z)$, as in [8, Section 8.1]

$$\begin{aligned} A_\varepsilon(t, z) &= \zeta_\varepsilon^{-1/2}(t, z) p_\varepsilon(t, z) + \zeta_\varepsilon^{1/2}(t, z) u_\varepsilon(t, z), \\ B_\varepsilon(t, z) &= -\zeta_\varepsilon^{-1/2}(t, z) p_\varepsilon(t, z) + \zeta_\varepsilon^{1/2}(t, z) u_\varepsilon(t, z). \end{aligned} \quad (3.1)$$

Here $u(\bar{T}t, \bar{L}z) = u_\varepsilon(t, z)$, with index ε indicating the dependence on the asymptotic parameter ε . The same notation applies to the pressure field $p_\varepsilon(t, z)$. The right going wave equals the impinging pulse at $z = 0$,

$$A_\varepsilon(t, 0) = f(t/\varepsilon^2). \quad (3.2)$$

At the order one scaled final range L we impose the condition

$$B_\varepsilon(t, L) = 0. \quad (3.3)$$

It follows from the truncation of the random medium in (2.12), justified by the causality of the wave equation, as explained in section 2.2. It says that up to time T we cannot observe waves backscattered beyond the range L .

3.1. The wave front and the random travel time. We define the wave front at range z as the right going wave observed in an ε^2 time window around the reference travel time $\tau = z/c_o$,

$$a_\varepsilon(\tau, s) = A_\varepsilon(t_\varepsilon(c_o\tau, s), c_o\tau). \quad (3.4)$$

Here

$$t_\varepsilon(c_o\tau, s) = \varepsilon^2 s + \int_0^{c_o\tau} \frac{dz'}{c_\varepsilon(t_\varepsilon(z', s), z')} = \tau + \varepsilon^2 s + \varepsilon^2 \mathcal{W}_\varepsilon(\tau, s), \quad (3.5)$$

is the travel time along the characteristics of the right going wave, as explained in section 4.1. The characteristics are parameterized by the range $z = c_o\tau$, and start at times $\varepsilon^2 s$ at $z = 0$, with s parameterizing the pulse. Note that $t_\varepsilon(c_o\tau, s)$ fluctuates randomly around the value $\tau + \varepsilon^2 s$, on the scale of the duration of the pulse, as modeled by the process

$$\mathcal{W}_\varepsilon(\tau, s) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}(s + \mathcal{W}_\varepsilon(u, s)), \frac{c_o u}{\varepsilon^\beta} \right). \quad (3.6)$$

The wave front (3.4) satisfies the following equation, as proved in section 4.2,

$$\begin{aligned} a_\varepsilon(\tau, s) = & f(s) - \frac{c_o^2}{8} \int_0^\tau d\tau' \mu_{\alpha, \beta}^+ \left[\frac{\tau' + \varepsilon^2 (s + \mathcal{W}_\varepsilon(\tau', s))}{\varepsilon^\alpha}, \frac{c_o \tau'}{\varepsilon^\beta} \right] \int_{Y_{\alpha, \beta}^\varepsilon(\tau', s)}^s dy \times \\ & \mu_{\alpha, \beta}^- \left[\frac{\tau' + \varepsilon^2 \left(\frac{y+s}{2} + \mathcal{W}_\varepsilon(\tau', s) \right)}{\varepsilon^\alpha}, \frac{c_o \tau' + \varepsilon^2 c_o (s-y)/2}{\varepsilon^\beta} \right] \times \\ & a_\varepsilon(\tau' + \varepsilon^2 (s-y)/2, \mathcal{S}_\varepsilon(y; \tau', s)) + O(\varepsilon), \end{aligned} \quad (3.7)$$

where the integration bound $Y_{\alpha, \beta}^\varepsilon(\tau', s) < s$ is given explicitly in section 4.2. The random processes $\mu_{\alpha, \beta}^\pm$ are

$$\mu_{\alpha, \beta}^\pm(t, z) = \varepsilon^{2-\beta} \mu_z(t, z) \pm \varepsilon^{2-\alpha} c_o^{-1} \mu_t(t, z), \quad (3.8)$$

with μ_z and μ_t the partial derivatives in z and t of μ . The mapping $\mathcal{S}_\varepsilon(y; \tau, s)$ is defined by the equation

$$\mathcal{S}_\varepsilon(y; \tau, s) + \mathcal{W}_\varepsilon(\tau, \mathcal{S}_\varepsilon(y; \tau, s)) = y + \mathcal{W}_\varepsilon(\tau, s) + \varepsilon \mathcal{R}_\varepsilon(y, \tau, s), \quad (3.9)$$

where \mathcal{R}_ε is bounded[§].

The complication in equation (3.7) stems from the possible dependence of the random fluctuations (3.6) of the travel time on s , the parameter of the pulse. This is why we need the mapping \mathcal{S}_ε , and the integration bound $Y_{\alpha, \beta}^\varepsilon$. To obtain an explicit characterization of the wave front, we consider two scaling regimes:

Regime 1 defined by $\alpha \leq 1$ and $\beta = 2$, where the medium is changing slowly.

Regime 2 defined by $\alpha = 2$ and $\beta \leq 1$, where the medium is changing rapidly.

The random travel time fluctuations are independent of s in these regimes[¶] as stated in the next lemma, proved in Appendix C. The implication is that we can set $Y_{\alpha, \beta}^\varepsilon = 0$

[§]The process \mathcal{R}_ε is defined explicitly later, in section 4.2.

[¶]The results can be extended to $\alpha < 2$ for regime 1 and $\beta < 2$ for regime 2, but the analysis is much more involved and will be presented in a different publication.

and $\mathcal{S}_\varepsilon(y; \tau, s) \xrightarrow{\varepsilon \rightarrow 0} y$, where the convergence is in probability. Equation (3.7) simplifies, and we can analyze explicitly its solution to obtain the pulse stabilization results described in the next section.

The analysis is based on the following mixing conditions satisfied by the process μ . They can be derived from the ϕ -mixing assumption on μ , with mixing rate $(\Delta z)^{-d}$, as shown in appendix B. There exists a deterministic constant C and a real number $d > 2$ such that

$$\mathbb{E} \left[\mu(t, z + \Delta z) \middle| \mathcal{F}_z \right] \leq C(\Delta z)^{-d}, \quad (3.10)$$

for all z, t and $\Delta z > 0$, where \mathcal{F}_z is the filtration associated to μ . Moreover, suppose that z_1 and z_2 satisfy $|z_1 - z| \geq \Delta z$ and $|z_2 - z| \geq \Delta z$. Then, for all z, t and t' we have

$$\left| \mathbb{E} \left[\mu(t, z_1) \mu(t', z_2) \middle| \mathcal{F}_z \right] - \mathbb{E} [\mu(t, z_1) \mu(t', z_2)] \right| \leq C(\Delta z)^{-d}, \quad (3.11)$$

Similar conditions are assumed for the derivatives μ_t and μ_z of the process, as well.

LEMMA 3.1. *Under the mixing assumption (3.10) for μ , and similar for μ_t , we have the following results: In regime 1*

$$\mathcal{W}_\varepsilon(\tau, s) = W_\varepsilon(\tau) + q_\varepsilon(c_o \tau, s),$$

where

$$W_\varepsilon(\tau) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left(\frac{u}{\varepsilon^\alpha}, \frac{c_0 u}{\varepsilon^2} \right), \quad (3.12)$$

and the residual q_ε satisfies

$$\mathbb{P} \left(\sup_{s \in [0, S], \tau \in [0, T]} |q_\varepsilon(c_o \tau, s)| \geq c \varepsilon^{(2-\alpha-\theta)/2} \right) \leq C \varepsilon^\theta, \quad (3.13)$$

for ε independent constants c and C , and any $\theta \in (0, 2 - \alpha)$. Moreover,

$$\mathbb{P} \left(\sup_{s \in [0, S], \tau \in [0, T]} |\partial_s \mathcal{W}_\varepsilon(\tau, s)| \geq c \varepsilon^{(2-\alpha-\theta)/2} \right) \leq C \varepsilon^\theta. \quad (3.14)$$

The result is similar in regime 2, under the corresponding mixing assumptions on μ_z , with α replaced by β and

$$W_\varepsilon(\tau) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left(\frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^\beta} \right) - \int_0^\tau du \mu^2 \left(\frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^\beta} \right). \quad (3.15)$$

Thus, in regimes 1 and 2 the random travel time fluctuations are described by the random process $W_\varepsilon(\tau)$. Under the mixing assumptions (3.10) and (3.11), we find [11] that $W_\varepsilon(\tau)$ in equation (3.12) converges in distribution to the Gaussian process

$$W(\tau) = h\tau + \mathcal{D}\mathcal{B}(\tau), \quad (3.16)$$

where $\mathcal{B}(\tau)$ is standard Brownian motion. The diffusion coefficient \mathcal{D} can be calculated from the variance

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\frac{1}{\varepsilon} \int_0^\tau du \mu \left(\frac{u}{\varepsilon^\alpha}, \frac{c_0 u}{\varepsilon^2} \right) \right]^2 = \tau \int_{-\infty}^\infty dt \Phi(0, c_o t) =: \tau \mathcal{D}^2 \quad (3.17)$$

in regime 1, and

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\frac{1}{\varepsilon} \int_0^\tau du \mu \left(\frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^\beta} \right) \right]^2 = \tau \int_{-\infty}^\infty dt \Phi(t, 0) =: \tau \mathcal{D}^2 \quad (3.18)$$

in regime 2. The drift h is obtained from the limit of the expectation of $W_\varepsilon(\tau)$. It is $h = 0$ in regime 1, and in regime 2 we get $h = -1$, because

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} [W_\varepsilon(\tau)] = - \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_0^\tau du \mu^2 \left(\frac{u}{\varepsilon^2}, \frac{c_0 u}{\varepsilon^\beta} \right) \right] = -\tau. \quad (3.19)$$

3.2. Pulse stabilization. The integral equation (3.7) of the wave front $a_\varepsilon(\tau, s)$ simplifies in regimes 1-2, with $Y_{\alpha, \beta}^\varepsilon$ replaced by 0 and $\mathcal{S}_\varepsilon(y; \tau, s)$ replaced by y . Moreover, $a_\varepsilon(\tau, s)$ converges to a deterministic function $\bar{a}(\tau, s)$, as stated next.

THEOREM 3.2. *Assume regime 1 or 2. The wave front $a_\varepsilon(\tau, s)$ converges in probability as $\varepsilon \rightarrow 0$, in the space of continuous functions of $(\tau, s) \in [0, T] \times [0, S]$, to the deterministic solution $\bar{a}(\tau, s)$ of the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial \tau} \bar{a}(\tau, s) &= -\frac{c_o^2}{8} \int_0^s dy \bar{a}(\tau, y) \Psi \left(\frac{s-y}{2} \right), \quad \tau > 0, \\ \bar{a}(0, s) &= f(s), \end{aligned} \quad (3.20)$$

with integral kernel

$$\Psi(s) = \begin{cases} -\partial_z^2 \Phi(0, c_o s) & \text{in regime 1,} \\ c_o^{-2} \partial_t^2 \Phi(s, 0) & \text{in regime 2.} \end{cases} \quad (3.21)$$

The limit is given explicitly by

$$\bar{a}(\tau, s) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} \hat{f}(\omega) \exp \left[-i\omega s - \frac{c_o^2 \tau}{4} \int_0^\infty du \Psi(u) e^{2i\omega u} \right]. \quad (3.22)$$

The convergence of a_ε to \bar{a} , the solution of (3.20), is derived in section 4.3. We assume stationary smooth medium fluctuations as explained in section 2.1, moreover, mixing conditions in (3.10) and (3.11) in Regime 1 and correspondingly in Regime 2.

To obtain (3.22) note that the right hand side in (3.20) is a convolution in y . Indeed, let us change variables in (3.20) as $y = s - u$, for $u \in [0, s]$, and extend the integral over u to infinity

$$\frac{\partial}{\partial \tau} \bar{a}(\tau, s) = -\frac{c_o^2}{8} \int_0^\infty du \bar{a}(\tau, s-u) \Psi \left(\frac{u}{2} \right), \quad (3.23)$$

because $\bar{a}(\tau, s-u) = 0$ for $s-u < 0$, by the compact support of the pulse in \mathbb{R}^+ . The Fourier transform $\widehat{\bar{a}}(\tau, \omega)$ with respect to s satisfies

$$\frac{\partial \widehat{\bar{a}}(\tau, \omega)}{\partial \tau} = -\frac{c_o^2}{4} \widehat{\bar{a}}(\tau, \omega) \int_0^\infty du \Psi(u) e^{2i\omega u}, \quad \tau > 0,$$

with initial condition $\widehat{\bar{a}}(0, \omega) = \widehat{f}(\omega)$. This can be solved explicitly and (3.22) follows by inverting the Fourier transform.

3.3. Interpretation of the results. Theorem 3.2 states that when we observe the right going wave at range z around the random travel time $t_\varepsilon(z, s)$, in a time interval equal to the duration of the pulse, the result is deterministic. This phenomenon is called *pulse stabilization*. Note however that the pulse shape is not the same as the emitted $f(s)$, like it would be in the homogeneous medium. The pulse is deformed, as modeled by the convolution kernel

$$\mathcal{K}_f(z, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega s - \omega^2 \gamma(\omega) z}, \quad z = c_o \tau, \quad (3.24)$$

where we introduced the notation

$$\gamma(\omega) = \frac{c_o}{4\omega^2} \int_0^\infty du \Psi(u) e^{2i\omega u}. \quad (3.25)$$

When we observe the wave at range z in a time window centered at the reference time $\tau = z/c_o$, the result is no longer deterministic, due to the random component \mathcal{W}_ε of the travel time. We state it in the corollary below, which follows from Theorem 3.2 and Slutsky's theorem:

COROLLARY 3.3. *As $\varepsilon \rightarrow 0$,*

$$A_\varepsilon(\tau + \varepsilon^2 s, c_o \tau) \rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{f}(\omega) e^{-i\omega[s - h\tau - \mathcal{D}\mathcal{B}(\tau)] - c_o \tau \omega^2 \gamma(\omega)}, \quad (3.26)$$

where the convergence is in distribution, $\mathcal{B}(\tau)$ is standard Brownian motion and h and \mathcal{D} are as in (3.16).

3.3.1. Slowly time changing media. In regime 1 we have

$$\begin{aligned} \text{real}[\gamma(\omega)] &= \frac{c_o}{4\omega^2} \int_0^\infty du \Psi(u) \cos(2\omega u) = -\frac{1}{4\omega^2} \int_0^\infty dz \partial_z^2 \Phi(0, z) \cos\left(\frac{2\omega}{c_o} z\right) \\ &= \frac{1}{2c_o^2} \int_{-\infty}^\infty dz \Phi(0, z) \cos\left(\frac{2\omega}{c_o} z\right) = \gamma^{(c)}(\omega) \geq 0, \end{aligned} \quad (3.27)$$

where the second equality is from definition (3.21), the third equality is obtained with integration by parts, and the fourth equality is a definition. That $\gamma^{(c)}$ is non-negative follows by Bochner's theorem. The imaginary part of $\gamma(\omega)$ is obtained similarly

$$\begin{aligned} \text{imag}[\gamma(\omega)] &= \frac{c_o}{4\omega^2} \int_0^\infty du \Psi(u) \sin(2\omega u) = -\frac{1}{4\omega^2} \int_0^\infty dz \partial_z^2 \Phi(0, z) \sin\left(\frac{2\omega}{c_o} z\right) \\ &= \frac{1}{c_o^2} \int_0^\infty dz \Phi(0, z) \sin\left(\frac{2\omega}{c_o} z\right) - \frac{1}{2\omega c_o} = \gamma^{(s)}(\omega) - \frac{1}{2\omega c_o}. \end{aligned} \quad (3.28)$$

We recovered the pulse stabilization result [8, Proposition 8.1] in time independent media. The random changes in the medium, on the time scale $T_\nu = \varepsilon^\alpha T$ that is much larger than the pulse width $T_\mathcal{F}$ and the time T_ℓ of traversal of a correlation length, play no role in the leading behavior of the wave front. The result is as if the medium were time independent. The kernel

$$\mathcal{K}(z, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega s - z\omega^2 [\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega)]} = \mathcal{K}_f\left(z, s - \frac{z}{2c_o}\right), \quad (3.29)$$

is analyzed in detail in [8, 10]. It satisfies the initial value problem

$$\frac{\partial \mathcal{K}(z, s)}{\partial z} = \mathcal{L}\mathcal{K}(z, s) \quad \text{for } z > 0, \quad \text{and } \mathcal{K}(z = 0, s) = \delta(s), \quad (3.30)$$

with pseudo-differential operator \mathcal{L} describing the pulse deformation caused by scattering in the medium. We can decompose it in two parts, $\mathcal{L} = \mathcal{L}^{(c)} + \mathcal{L}^{(s)}$, where

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{L}^{(c)}\mathcal{K}(z, s)e^{i\omega s} ds &= -\omega^2\gamma^{(c)}(\omega) \int_{-\infty}^{\infty} \mathcal{K}(z, s)e^{i\omega s} ds, \\ \int_{-\infty}^{\infty} \mathcal{L}^{(s)}\mathcal{K}(z, s)e^{i\omega s} ds &= -i\omega^2\gamma^{(s)}(\omega) \int_{-\infty}^{\infty} \mathcal{K}(z, s)e^{i\omega s} ds. \end{aligned} \quad (3.31)$$

The first part $\mathcal{L}^{(c)}$ acts as an effective diffusion operator. It models the frequency-dependent attenuation over range z , at the rate $\omega^2\gamma^{(c)}(\omega)$. The second part $\mathcal{L}^{(s)}$ is an effective dispersion operator that preserves energy. The dispersive effect increases with z at rate $\omega^2\gamma^{(s)}(\omega)$. We can rewrite (3.31) in the time domain using (3.27)-(3.28)

$$\begin{aligned} \mathcal{L}\mathcal{K}(z, s) &= \left[\frac{1}{2c_o} \Phi\left(0, \frac{c_o s}{2}\right) \mathbf{1}_{[0, \infty)}(s) \right] * \left[\frac{\partial^2 \mathcal{K}}{\partial s^2}(z, s) \right] \\ &= \frac{1}{2c_o} \int_0^{\infty} \Phi\left(0, \frac{c_o \xi}{2}\right) \frac{\partial^2 \mathcal{K}}{\partial s^2}(z, s - \xi) d\xi. \end{aligned} \quad (3.32)$$

The indicator function $\mathbf{1}_{[0, \infty)}$ entails that if \mathcal{K} is vanishing for $s < s_0$, then $\mathcal{L}\mathcal{K}$ is also vanishing for $s < s_0$. That is to say, the operator \mathcal{L} preserves causality. The operator also satisfies the Kramers-Kronig relations [6, 13], as shown in [10].

Corollary 3.3 gives that $A_\varepsilon(\tau + \varepsilon^2 s, c\tau) \xrightarrow{\varepsilon \rightarrow 0} f_\tau(s - \frac{\tau}{2} - \mathcal{D}\mathcal{B}(\tau))$ in distribution, where $f_\tau(s) = f(s) \star \mathcal{K}(c\tau, s)$ is the deterministic pulse shape given by the convolution of $f(s)$ with the kernel $\mathcal{K}(z, s)$. The diffusion coefficient \mathcal{D} in front of the Brownian motion is $\mathcal{D} = \sqrt{2c_o\gamma^{(c)}(0)}$. Note that the result is the same as in [8, Proposition 8.1], except for the deterministic delay of $\tau/2$. This is because we have a different model of the random wave speed. In [8, Proposition 8.1] the denominator in (2.17) has a square root, and this results in a deterministic correction of the travel time that cancels the delay.

3.3.2. Rapidly time changing media. In regime 2 we have

$$\begin{aligned} \text{real}[\gamma(\omega)] &= \frac{c_o}{4\omega^2} \int_0^{\infty} du \Psi(u) \cos(2\omega u) = \frac{1}{4c_o\omega^2} \int_0^{\infty} \partial_t^2 \Phi(t, 0) \cos(2\omega t) \\ &= -\frac{1}{2c_o} \int_{-\infty}^{\infty} dt \Phi(t, 0) \cos(2\omega t) = \gamma^{(c)}(\omega) \leq 0, \end{aligned} \quad (3.33)$$

where the second equality is from definition (3.21), the third equality is obtained with integration by parts, and the fourth equality is a definition. Note that the real part of γ is now negative. The imaginary part of $\gamma(\omega)$ is obtained similarly

$$\begin{aligned} \text{imag}[\gamma(\omega)] &= \frac{c_o}{4\omega^2} \int_0^{\infty} du \Psi(u) \sin(2\omega u) = \frac{1}{4c_o\omega^2} \int_0^{\infty} dt \partial_t^2 \Phi(t, 0) \sin(2\omega t) \\ &= -\frac{1}{c_o} \int_0^{\infty} dt \Phi(t, 0) \sin(2\omega t) + \frac{1}{2\omega c_o} = \gamma^{(s)}(\omega) + \frac{1}{2\omega c_o}. \end{aligned} \quad (3.34)$$

As we did before, we define the pulse shaping kernel $\mathcal{K}(z, s)$ by

$$\mathcal{K}(z, s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega s - z\omega^2[\gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega)]} = \mathcal{K}_f \left(z, s + \frac{z}{2c_o} \right). \quad (3.35)$$

This is to be understood weakly, as the finite bandwidth makes the convolution $f_\tau(s) = f(s) \star_s \mathcal{K}(z, s)$ well defined. We obtain that

$$A_\varepsilon(\tau + \varepsilon^2 s, c\tau) \xrightarrow{\varepsilon \rightarrow 0} f_\tau \left(s + \frac{3\tau}{2} - \mathcal{DB}(\tau) \right), \quad (3.36)$$

in distribution. Unlike the result in slowly changing media where the pulse fades as it travels through the medium, in rapidly changing media the pulse is enhanced. This is because $\gamma^{(c)}(\omega)$ is negative and energy is not conserved in time dependent media. Thus, the random fluctuations of the wave speed in time and range have opposite effects. We illustrate this with a simple example in section 3.4. Before that, we draw an analogy between the pulse stabilization result in rapidly changing media and the random harmonic oscillator problem studied in [8, Section 7.5] and [12].

3.3.3. Analogy to the random harmonic oscillator problem. Let us suppose for a moment that there were no spatial fluctuations, just the temporal ones. We denote these fluctuations by $\tilde{\mu}$ to distinguish them from the time and range fluctuations in (2.17). We also change variables $z = c_o \mathcal{T}, t = Z/c_o$, and obtain a new wave equation for the displacement u ,

$$\frac{\partial^2 u(\mathcal{T}, Z)}{\partial Z^2} - \frac{\left[1 + \varepsilon \tilde{\mu} \left(\frac{Z/c_o}{\varepsilon^2} \right) \right]}{c_o^2} \frac{\partial^2 u(\mathcal{T}, Z)}{\partial \mathcal{T}^2} = 0, \quad Z > 0,$$

where

$$1 + \varepsilon \tilde{\mu} \left(\frac{Z/c_o}{\varepsilon^2} \right) = \left[1 + \varepsilon \mu \left(\frac{Z/c_o}{\varepsilon^2}, 0 \right) \right]^{-2}.$$

Before we had a two point boundary value problem for the right and left going waves. The right going wave amplitude was specified in terms of the impinging pulse $f(t/\varepsilon^2)$ by (3.2) at $z = 0$. The left going wave amplitude was set to zero at range L , using causality and the finite time T of observation. Now, after the change of variables, we have initial conditions at $Z = 0$, which corresponds to $t = 0$,

$$u(\mathcal{T}, Z = 0) = f(\mathcal{T}/\varepsilon^2)/(2\sqrt{\zeta_o}), \quad \frac{\partial u(\mathcal{T}, Z = 0)}{\partial Z} = f'(\mathcal{T}/\varepsilon^2)/(2\varepsilon^2 c_o \sqrt{\zeta_o}).$$

The first initial condition follows from the wave decomposition (3.1) evaluated at $Z = 0$ (i.e., $t = 0$), where the wave is right going with amplitude given by the pulse $f(\mathcal{T}/\varepsilon^2)$. The second condition follows from the same decomposition and the relation $\partial_z u(0, \mathcal{T}) = -1/(c_o \zeta_o) \partial_\tau p(0, \mathcal{T})$ obtained from the acoustic system of equations.

Fourier transforming in \mathcal{T} , we obtain that $\hat{u}_\varepsilon(\omega, Z) = \hat{u}(\frac{\omega}{\varepsilon^2}, Z)$ satisfies

$$\frac{\partial^2 \hat{u}_\varepsilon(\omega, Z)}{\partial Z^2} + \frac{\omega^2}{\varepsilon^4 c_o^2} \left[1 + \varepsilon \tilde{\mu} \left(\frac{Z/c_o}{\varepsilon^2} \right) \right] \hat{u}_\varepsilon(\omega, Z) = 0, \quad Z > 0, \quad (3.37)$$

with initial conditions

$$\hat{u}_\varepsilon(\omega, 0) = \frac{\hat{f}(\omega)}{2\sqrt{\zeta_o}}, \quad \frac{\partial \hat{u}_\varepsilon(\omega, 0)}{\partial Z} = -\frac{i\omega \hat{f}(\omega)}{2c_o \sqrt{\zeta_o}}.$$

This Cauchy problem is studied in [8, Section 7.5.3], for random harmonic oscillators. The rate corresponding to exponential growth in Z of the square root of the energy $|\hat{u}_\varepsilon|^2 + |\partial_Z \hat{u}_\varepsilon|^2$ is the Lyapunov exponent defined in [8, Proposition 7.9]. It coincides with the rate $\omega^2 |\gamma^{(c)}|$ predicted by our analysis.

3.4. Example of a medium in translational motion. The integral equation (3.7) of the wave front is complicated when $\alpha = \beta = 2$, because of the strong dependence of the random travel time fluctuations on the starting point s of the characteristics. We do not have an explicit description of the wave front for an arbitrary model μ of the fluctuations, but there are models where the problem simplifies. We illustrate here one of them, corresponding to media in uniform translational motion

$$\mu\left(\frac{t}{\varepsilon^2}, \frac{z}{\varepsilon^2}\right) = \eta\left(\frac{z - vt}{\varepsilon^2}\right), \quad (3.38)$$

where η is a mean zero stationary random process with autocorrelation Φ . The process η is as smooth as before and with finite dependence range. The speed v may be positive or negative, but it satisfies the inequality $|v| < c_o$.

The random travel time fluctuations (3.6) are

$$\mathcal{W}_\varepsilon(\tau, s) = \frac{1}{\varepsilon} \int_0^\tau du \eta\left(\frac{(c_o - v)u}{\varepsilon^2} - vs - v\mathcal{W}_\varepsilon(u, s)\right), \quad (3.39)$$

and changing variables $\zeta = (c_o - v)u - \varepsilon^2 vs - \varepsilon^2 v\mathcal{W}_\varepsilon(u, s)$, with ζ lying in the interval $\zeta \in [O(\varepsilon^2), (c_o - v)\tau + O(\varepsilon^2)]$, we obtain

$$\mathcal{W}_\varepsilon(\tau, s) = \frac{1}{\varepsilon(c_o - v)} \int_0^{(c_o - v)\tau} d\zeta \eta(\zeta/\varepsilon^2) + O(\varepsilon). \quad (3.40)$$

This is independent of s to leading order, and \mathcal{W}_ε converges in distribution, as $\varepsilon \rightarrow 0$, to the diffusion process $\mathcal{W}(\tau) = \mathcal{D}\mathcal{B}(\tau)$, where $\mathcal{B}(\tau)$ is standard Brownian motion and $\mathcal{D} = \sqrt{\hat{\Phi}(0)/(c_o - v)}$. Thus, we can set $Y_{\alpha, \beta}^\varepsilon = 0$ and $\mathcal{S}_\varepsilon(y; \tau, s) = y$ in (3.7), and the random processes (3.8) are

$$\mu_{\alpha, \beta}^\pm(t, z) = (1 \mp v/c_o)\eta'(z - vt). \quad (3.41)$$

The pulse stabilization results in Theorem 3.2 and Corollary 3.3 hold as stated, with

$$\Psi(s) = \left[(v/c_o)^2 - 1\right]\Phi''((c_o + v)s), \quad (3.42)$$

and the parameter $\gamma(\omega)$ in the pulse shaping kernel is

$$\begin{aligned} \gamma(\omega) &= \frac{(v^2 - c_o^2)}{4\omega^2 c_o} \int_0^\infty du \Phi''((c_o + v)u) e^{2i\omega u} \\ &= i \frac{(v - c_o)}{2\omega c_o (c_o + v)} + \gamma^{(c)}(\omega) + i\gamma^{(s)}(\omega). \end{aligned} \quad (3.43)$$

Here we used integration by parts and the normalization $\Phi(0) = 1$, and let

$$\gamma^{(c)}(\omega) = \frac{(c_o - v)}{2c_o(c_o + v)^2} \hat{\Phi}\left(\frac{2\omega}{c_o + v}\right) \quad (3.44)$$

$$\gamma^{(s)}(\omega) = \frac{(c_o - v)}{c_o(c_o + v)^2} \int_0^\infty dz \Phi(z) \sin[2\omega z/(c_o + v)]. \quad (3.45)$$

These results show that as $v \uparrow c_o$ the transformation of the pulse shape becomes small, as if the pulse had propagated only through a thin section of the random medium. This is because the right propagating wave component travels with a speed only slightly larger than the medium. However, the random travel time correction becomes large since the averaging becomes “less effective”. The diffusion becomes small also in the limit $v \downarrow -c_o$, due to the assumed rapid decay of the power spectral density of the fluctuations, while the dispersive effect becomes strong.

4. Proof of results. The proof builds on the approach in [8, Section 8.1]. See also [3, 4]. We use random travel time coordinates to derive in section 4.2 a time domain integral equation for the wave front. The coordinates are motivated by the method of characteristics applied to the first order system of partial differential equations satisfied by the right and left going waves, as described in section 4.1. The integral equation for the wave front is (3.7) and holds for all $\alpha, \beta \in (0, 2]$. However, it simplifies in regimes 1-2, where the random fluctuations of the travel time are independent of the characteristics in the limit $\varepsilon \rightarrow 0$. The simplification is explained below and it leads to the explicit expression of the wave front stated in Theorem 3.2, as shown in section 4.3.

4.1. The wave decomposition and random travel time coordinates. The right and left going waves $A_\varepsilon(t, z)$ and $B_\varepsilon(t, z)$ defined by (3.1) satisfy the following system of first order partial differential equations derived in appendix A

$$\frac{1}{c_\varepsilon(t, z)} \frac{\partial A_\varepsilon(t, z)}{\partial t} + \frac{\partial A_\varepsilon(t, z)}{\partial z} = \frac{1}{\varepsilon} [M_\varepsilon(t, z) + N_\varepsilon(t, z)] B_\varepsilon(t, z), \quad (4.1)$$

$$\frac{1}{c_\varepsilon(t, z)} \frac{\partial B_\varepsilon(t, z)}{\partial t} - \frac{\partial B_\varepsilon(t, z)}{\partial z} = \frac{1}{\varepsilon} [-M_\varepsilon(t, z) + N_\varepsilon(t, z)] A_\varepsilon(t, z), \quad (4.2)$$

with initial condition (3.2) for A_ε and final condition (3.3) for B_ε . Here

$$M_\varepsilon(t, z) = -\frac{\varepsilon^{2-\beta} \mu_z \left(\frac{t}{\varepsilon^\alpha}, \frac{z}{\varepsilon^\beta} \right)}{2 \left[1 + \varepsilon \mu \left(\frac{t}{\varepsilon^\alpha}, \frac{z}{\varepsilon^\beta} \right) \right]}, \quad N_\varepsilon(t, z) = -\frac{\varepsilon^{2-\alpha} \mu_t \left(\frac{t}{\varepsilon^\alpha}, \frac{z}{\varepsilon^\beta} \right)}{2c_o \left[1 + \varepsilon \mu \left(\frac{t}{\varepsilon^\alpha}, \frac{z}{\varepsilon^\beta} \right) \right]^{1/2}}, \quad (4.3)$$

where μ_z and μ_t are the partial derivatives of μ in z and t .

We solve the first order system (4.1)-(4.2) using the method of characteristics. The characteristics of the right going waves are the curves $(t_\varepsilon(z, s), z)$ in the (t, z) plane, where $t_\varepsilon(z, s)$ is the travel time (3.5), and s parameterizes the impinging pulse at $z = 0$, giving the initial condition

$$A_\varepsilon(t = \varepsilon^2 s, z = 0) = f(s), \quad s \in [0, S]. \quad (4.4)$$

The right going wave observed at the random travel time is obtained by integrating equation (4.1) along the characteristics

$$A_\varepsilon(t_\varepsilon(z, s), z) = f(s) + \frac{1}{\varepsilon} \int_0^z dz' G_\varepsilon(t_\varepsilon(z', s), z'), \quad (4.5)$$

where

$$G_\varepsilon(t, z) = [M_\varepsilon(t, z) + N_\varepsilon(t, z)] B_\varepsilon(t, z). \quad (4.6)$$

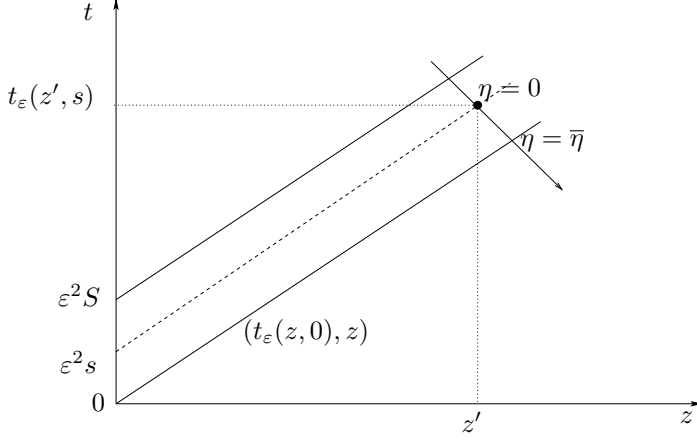


FIG. 4.1. The characteristics curves $(t_\varepsilon(z, s), z)$ of the right going waves start at times $\varepsilon^2 s$ on the initial curve $z = 0$, with $s \in [0, S]$ parameterizing the pulse. We draw the characteristics approximately, as straight lines of slope $1/c_o$, but we recall that $t_\varepsilon(z, s)$ has random fluctuations of order ε^2 . The characteristic of the left going wave passing through the point $(t_\varepsilon(z', s), z')$ is parameterized by $-\eta$, with $\eta = 0$ at the point of intersection. The front of the pulse is on the characteristic curve $(t_\varepsilon(z, 0), z)$, stemming from $s = 0$ at $z = 0$.

To determine the left going wave $B_\varepsilon(t_\varepsilon(z', s), z')$, we integrate (4.2) along its characteristic passing through $(t_\varepsilon(z', s), z')$. As illustrated in Figure 4.1, we parameterize this characteristic by $-\eta$, starting from the point of intersection. We have

$$B_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta; z', s), z' + \eta) = B_\varepsilon(t_\varepsilon(z', s), z') - \frac{1}{\varepsilon} \int_0^\eta d\eta' Q_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta'; z', s), z' + \eta'), \quad (4.7)$$

where

$$Q_\varepsilon(t, z) = [-M_\varepsilon(t, z) + N_\varepsilon(t, z)] A_\varepsilon(t, z), \quad (4.8)$$

and

$$\tilde{t}_\varepsilon(\eta; z', s) = - \int_0^\eta \frac{d\eta'}{c_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta'; z', s), z' + \eta')}. \quad (4.9)$$

By causality, the left going wave is zero ahead of the front given by the curve $(t_\varepsilon(z, 0), z)$ for $z > 0$. We denote η at the front by $\bar{\eta}$. It is defined implicitly by

$$\tilde{t}_\varepsilon(\bar{\eta}; z', s) = t_\varepsilon(z' + \bar{\eta}, 0) - t_\varepsilon(z', s) = \frac{\bar{\eta}}{c_o} - \varepsilon^2 \left[s + \mathcal{W}_\varepsilon\left(\frac{z'}{c_o}, s\right) - \mathcal{W}_\varepsilon\left(\frac{z' + \bar{\eta}}{c_o}, 0\right) \right].$$

Now substitute this in (4.9) and use the model (2.17) of the wave speed to obtain

$$\frac{2\bar{\eta}}{c_o} + \varepsilon \int_0^{\bar{\eta}} \frac{d\eta'}{c_o} \mu\left(\frac{t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta'; z', s)}{\varepsilon^\alpha}, \frac{z' + \eta'}{\varepsilon^\beta}\right) = \varepsilon^2 s + \varepsilon^2 \left[\mathcal{W}_\varepsilon\left(\frac{z'}{c_o}, s\right) - \mathcal{W}_\varepsilon\left(\frac{z' + \bar{\eta}}{c_o}, 0\right) \right].$$

Recalling that μ is bounded, and using definition (3.6) of \mathcal{W}_ε and the mean value theorem we have

$$\bar{\eta} \left[\frac{2}{c_o} + O(\varepsilon) \right] = \varepsilon^2 s + \varepsilon^2 \left[\mathcal{W}_\varepsilon \left(\frac{z'}{c_o}, s \right) - \mathcal{W}_\varepsilon \left(\frac{z'}{c_o}, 0 \right) \right].$$

Here $O(\varepsilon)$ denotes a term that is bounded by $C\varepsilon$, for an ε independent constant C . Thus, we conclude that

$$\bar{\eta} = \frac{\varepsilon^2 c_o}{2} \left[s + \mathcal{W}_\varepsilon \left(\frac{z'}{c_o}, s \right) - \mathcal{W}_\varepsilon \left(\frac{z'}{c_o}, 0 \right) \right] + O(\varepsilon^3). \quad (4.10)$$

Lemma 3.1 states that in regimes 1-2 the random process $\mathcal{W}_\varepsilon(\tau, s)$ is independent of s to leading order, so $\bar{\eta} \approx \varepsilon^2 c_o s / 2$. In the general case $\alpha, \beta \in (0, 2]$ we may have $\mathcal{W}_\varepsilon(\tau, s)$ depending on s , so $\bar{\eta}$ may be random. Nevertheless, $\bar{\eta}$ is guaranteed to be positive to leading order, because the characteristic stemming from $\varepsilon^2 s$ at $z = 0$ cannot intersect^{||} the front of the pulse, the characteristic stemming from the origin.

The left going wave at the point $(t_\varepsilon(z', s), z')$ of intersection with the right going characteristic is

$$B_\varepsilon(t_\varepsilon(z', s), z') = \frac{1}{\varepsilon} \int_0^{\bar{\eta}} d\eta' Q_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta'; z', s), z' + \eta'). \quad (4.11)$$

It follows by setting $\eta = \bar{\eta}$ in (4.7), because the left hand side is the left going wave which vanishes at the wave front.

4.2. The integral equation. The integral equation for the wave front is obtained by substituting (4.11) in (4.5). We write it in more convenient form by changing variables in (4.11) as $\eta' = \varepsilon^2 c_o (s - y) / 2$, with y satisfying

$$Y_{\alpha, \beta}^\varepsilon(z', s) \leq y \leq s, \quad \text{for } Y_{\alpha, \beta}^\varepsilon(z', s) = \mathcal{W}_\varepsilon(z'/c_o, 0) - \mathcal{W}_\varepsilon(z'/c_o, s). \quad (4.12)$$

The equation for the left going wave becomes

$$B_\varepsilon(t_\varepsilon(z', s), z') = \frac{\varepsilon c_o}{2} \int_{Y_{\alpha, \beta}^\varepsilon(z', s)}^s dy Q_\varepsilon \left[t_\varepsilon(z', s) + \tilde{t}_\varepsilon \left(\frac{\varepsilon^2 c_o (s - y)}{2}; z', s \right), z' + \frac{\varepsilon^2 c_o (s - y)}{2} \right], \quad (4.13)$$

and substituting it in (4.5) and using definition (4.8) of Q_ε , we get the following equation for the right going wave

$$\begin{aligned} A_\varepsilon[t_\varepsilon(z, s), z] &= f(s) - \frac{c_o}{2} \int_0^z dz' (M_\varepsilon + N_\varepsilon)(t_\varepsilon(z', s), z') \times \\ &\quad \int_{Y_{\alpha, \beta}^\varepsilon(z', s)}^s dy A_\varepsilon \left[t_\varepsilon(z', s) + \tilde{t}_\varepsilon \left(\frac{\varepsilon^2 c_o (s - y)}{2}; z', s \right), z' + \frac{\varepsilon^2 c_o (s - y)}{2} \right] \times \\ &\quad (M_\varepsilon - N_\varepsilon) \left[t_\varepsilon(z', s) + \tilde{t}_\varepsilon \left(\frac{\varepsilon^2 c_o (s - y)}{2}; z', s \right), z' + \frac{\varepsilon^2 c_o (s - y)}{2} \right]. \end{aligned} \quad (4.14)$$

^{||}The characteristics do not intersect because they are solutions of a flow (ordinary differential equation) problem that is uniquely solvable [7, Section 3.2.5.a].

Next we rewrite this integral equation in terms of the right propagating wave in (3.4) expressed in characteristic coordinates and to do so we introduce below a coordinate mapping. First we obtain from (4.9) that

$$\begin{aligned}\tilde{t}_\varepsilon(\varepsilon^2 c_o(s-y)/2; z', s) &= - \int_0^{\frac{\varepsilon^2 c_o(s-y)}{2}} \frac{d\eta'}{c_\varepsilon(t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\eta'; z', s), z' + \eta')} \\ &= -\varepsilon^2(s-y)/2 + \varepsilon^3 R_\varepsilon(y, z'/c_o, s),\end{aligned}\quad (4.15)$$

where we used definition (2.17) of c_ε , changed variables $\eta' = \varepsilon^2 \zeta$, and introduced the remainder

$$R_\varepsilon(y, z'/c_o, s) = - \int_0^{\frac{c_o(s-y)}{2}} d\zeta \mu\left(\frac{t_\varepsilon(z', s) + \tilde{t}_\varepsilon(\varepsilon^2 \zeta; z', s)}{\varepsilon^\alpha}, \frac{z' + \varepsilon^2 \zeta}{\varepsilon^\beta}\right),$$

with both R_ε and $\partial_y R_\varepsilon$ bounded. Recalling definition (3.5) of t_ε , we get

$$\begin{aligned}t_\varepsilon(z', s) + \tilde{t}_\varepsilon\left(\frac{\varepsilon^2 c_o(s-y)}{2}; z', s\right) &= \frac{z'}{c_o} + \frac{\varepsilon^2(s-y)}{2} + \\ &\quad \varepsilon^2 \left[y + \mathcal{W}_\varepsilon\left(\frac{z'}{c_o}, s\right) + \varepsilon R_\varepsilon(y, z'/c_o, s) \right],\end{aligned}\quad (4.16)$$

and rewrite the right hand side in terms of the mapping $\mathcal{S}_\varepsilon(y; \tau, s)$ defined by

$$\mathcal{S}_\varepsilon\left(y; \frac{z'}{c_o}, s\right) = y + \mathcal{W}_\varepsilon\left(\frac{z'}{c_o}, s\right) - \mathcal{W}_\varepsilon\left(\frac{z'}{c_o} + \frac{\varepsilon^2(s-y)}{2}, \mathcal{S}_\varepsilon\left(y; \frac{z'}{c_o}, s\right)\right) + \varepsilon R_\varepsilon(y, z'/c_o, s).\quad (4.17)$$

Since $\varepsilon^2(s-y)\partial_\tau \mathcal{W}_\varepsilon(\tau, s) = O(\varepsilon)$, uniformly in τ, s and y , we see that this is the same as definition (3.9), with \mathcal{R}_ε equal to R_ε plus an order one correction coming from expanding \mathcal{W}_ε around $\tau' = z'/c_o$. Using (4.17) in (4.16) we obtain that

$$\begin{aligned}A_\varepsilon\left[t_\varepsilon(z', s) + \tilde{t}_\varepsilon\left(\frac{\varepsilon^2 c_o(s-y)}{2}; z', s\right), z' + \frac{\varepsilon^2 c_o(s-y)}{2}\right] &= \\ &= a_\varepsilon\left(z'/c_o + \frac{\varepsilon^2(s-y)}{2}, \mathcal{S}_\varepsilon(y; z'/c_o, s)\right),\end{aligned}$$

and equation (4.14) becomes

$$\begin{aligned}a_\varepsilon(\tau, s) &= f(s) - \frac{c_o^2}{2} \int_0^\tau d\tau' (M_\varepsilon + N_\varepsilon) (t_\varepsilon(c_o\tau', s), c_o\tau') \times \\ &\quad \int_{Y_{\alpha, \beta}^\varepsilon(c_o\tau', s)}^s dy a_\varepsilon(\tau' + \varepsilon^2(s-y)/2, \mathcal{S}_\varepsilon(y; \tau', s)) \times \\ &\quad (M_\varepsilon - N_\varepsilon) \left[t_\varepsilon(c_o\tau', s) + \tilde{t}_\varepsilon\left(\frac{\varepsilon^2 c_o(s-y)}{2}; z', s\right), c_o\tau' + \frac{\varepsilon^2 c_o(s-y)}{2} \right].\end{aligned}\quad (4.18)$$

Equation (3.7) follows by substitution of definitions (4.3) of M_ε and N_ε .

It remains to show that (4.18) simplifies in regimes 1 and 2. Note first that in view of Lemma 3.1, $Y_{\alpha, \beta}^\varepsilon$ is smaller than ε to some positive power, on an event E with probability $\mathbb{P}(E) > 1 - O(\varepsilon^\theta)$. Recall also that the bound $Y_{\alpha, \beta}^\varepsilon$ on y corresponds to the wavefront, meaning that for $y < Y_{\alpha, \beta}^\varepsilon$ the contribution to the integral is zero. Thus,

we can extend the y integral to the interval $[-S, s]$ without making any mistake. In view of Lemma 3.1 and Eq. (4.17) we also have that:

$$\mathbb{P}\left(\sup_{\tau \in [0, T], 0 \leq y \leq s \leq S} |\partial_y \mathcal{S}^\varepsilon(y; \tau, s) - 1| \geq c\varepsilon\right) \leq C\varepsilon^\theta. \quad (4.19)$$

Then we get via a change of variables in (4.18) that on an event with probability $1 - O(\varepsilon^\theta)$,

$$\begin{aligned} a_\varepsilon(\tau, s) &= f(s) - \frac{c_o^2}{2} \int_0^\tau d\tau' (M_\varepsilon + N_\varepsilon) (t_\varepsilon(c_o\tau', s), c_o\tau') \times \\ &\quad \int_{-S}^s dy a_\varepsilon(\tau' + \varepsilon^2(s-y)/2, y) \times \\ &\quad (M_\varepsilon - N_\varepsilon) \left[t_\varepsilon(c_o\tau', s) + \tilde{t}_\varepsilon\left(\frac{\varepsilon^2 c_o(s-y)}{2}; z', s\right), c_o\tau' + \frac{\varepsilon^2 c_o(s-y)}{2} \right] \times \\ &\quad (1 + \varepsilon h_\varepsilon(y, \tau', s)), \end{aligned} \quad (4.20)$$

with some bounded h_ε .

We show next that $\partial_\tau a_\varepsilon$ is bounded, so we can simplify the first argument of a_ε in (4.20). When taking the partial derivative with respect to τ we see that we can bound $\partial_\tau a_\varepsilon$ in terms of a_ε . To bound the latter, we recall that M_ε and N_ε are uniformly bounded. We then get

$$\|a_\varepsilon(\cdot, s)\|_\infty \leq \|f\|_\infty + C \int_{-S}^s dy \|a_\varepsilon(\cdot, y)\|_\infty,$$

where we introduced the notation

$$\|a_\varepsilon(\cdot, s)\|_\infty = \sup_\tau |a_\varepsilon(\tau, s)|.$$

The uniform boundedness of a_ε , and therefore of $\partial_\tau a_\varepsilon$, follows from Gronwall's inequality. We thus find, using also the smoothness of μ , that on an event with probability $1 - O(\varepsilon^\theta)$ we can simplify the equation (3.7) of the wave front as

$$a_\varepsilon(\tau, s) = f(s) + \int_0^\tau d\tau' \int_{-S}^s dy \mathcal{G}_\varepsilon(\tau', s, y) a_\varepsilon(\tau', y) + O(\varepsilon), \quad (4.21)$$

where

$$\begin{aligned} \mathcal{G}_\varepsilon(\tau', s, y) &:= -\frac{c_o^2}{8} \mu_{\alpha, \beta}^+ \left[\frac{\tau' + \varepsilon^2(s + \mathcal{W}_\varepsilon(\tau', s))}{\varepsilon^\alpha}, \frac{c_o\tau'}{\varepsilon^\beta} \right] \times \\ &\quad \mu_{\alpha, \beta}^- \left[\frac{\tau' + \varepsilon^2\left(\frac{y+s}{2} + \mathcal{W}_\varepsilon(\tau', s)\right)}{\varepsilon^\alpha}, \frac{c_o\tau' + \varepsilon^2 c_o(s-y)/2}{\varepsilon^\beta} \right], \end{aligned} \quad (4.22)$$

with $\mu_{\alpha, \beta}^\pm$ defined in (3.8).

4.3. Stochastic averaging. To show that $a_\varepsilon \rightarrow \bar{a}$ as stated in Theorem 3.2, it suffices to show that

$$\sup_{\tau \leq T} \|a_\varepsilon(\tau, \cdot) - \bar{a}(\tau, \cdot)\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.23)$$

in probability. We give the proof in regime 1, with $\alpha < 1$ and $\beta = 2$, but the analysis in regime 2 is very similar.

4.3.1. The linear integral operators. Let us rewrite equation (4.21) as

$$a_\varepsilon(\tau, s) = f(s) + \int_0^\tau du [\mathcal{L}_\varepsilon(u) a_\varepsilon(u, \cdot)](s), \quad (4.24)$$

using the linear operator $\mathcal{L}_\varepsilon(u)$ defined by

$$[\mathcal{L}_\varepsilon(u)\varphi](s) = \int_{-S}^s dy \mathcal{G}_\varepsilon(u, s, y)\varphi(y), \quad \forall \varphi \in \mathcal{C}_{[-S, S]}, \quad (4.25)$$

on the space $\mathcal{C}_{[-S, S]}$ of continuous functions on $[-S, S]$. Here \mathcal{G}_ε is defined by (4.22) for $\tau' = u$, and $\mu_{\alpha, 2}^\pm$ is obtained from definition (3.8)

$$\mu_{\alpha, 2}^\pm(t, z) = \mu_z(t, z) \pm \varepsilon^{2-\alpha} c_o^{-1} \mu_t(t, z). \quad (4.26)$$

Since μ_z is bounded, we conclude that $\mathcal{L}_\varepsilon(u)$ is Lipschitz continuous (i.e., bounded) with deterministic constant $C_\mathcal{L}$ that is independent of ε and u ,

$$\|\mathcal{L}_\varepsilon(u)\varphi\|_\infty \leq C_\mathcal{L} \|\varphi\|_\infty, \quad \forall \varphi \in \mathcal{C}_{[-S, S]}. \quad (4.27)$$

Similarly, we write the equation satisfied by the limit \bar{a} as

$$\bar{a}(\tau, s) = f(s) + \int_0^\tau du [\bar{\mathcal{L}}\bar{a}(u, \cdot)](s), \quad (4.28)$$

using the bounded linear operator $\bar{\mathcal{L}} : \mathcal{C}_{[-S, S]} \rightarrow \mathcal{C}_{[-S, S]}$, defined by

$$[\bar{\mathcal{L}}\varphi](s) = -\frac{c_o^2}{8} \int_{-S}^s dy \varphi(y) \Psi\left(\frac{s-y}{2}\right), \quad \forall \varphi \in \mathcal{C}_{[-S, S]}, \quad (4.29)$$

where

$$\Psi(s) = \mathbb{E} \{\mu_z(t, z)\mu_z(t, z + c_o s)\} = -\partial_z^2 \Phi(0, c_o s). \quad (4.30)$$

Note that (4.29) is the same as the expectation of (4.25) up to a term of order $\varepsilon^{2-\alpha}$. We also remark that we could have replaced the lower integration bound $-S$ in (4.29) by 0 due to the support of the initial condition f in (4.28).

4.3.2. Averaging. We now show that the limit (4.23) holds. We have

$$a_\varepsilon(\tau, s) - \bar{a}(\tau, s) = g_\varepsilon(\tau, s) + \int_0^\tau du [\mathcal{L}_\varepsilon(u) (a_\varepsilon(u, \cdot) - \bar{a}(u, \cdot))](s), \quad (4.31)$$

where

$$g_\varepsilon(\tau, s) = \int_0^\tau du [(\mathcal{L}_\varepsilon(u) - \bar{\mathcal{L}})\bar{a}(u, \cdot)](s). \quad (4.32)$$

Recalling that $\mathcal{L}_\varepsilon(u)$ is bounded, and using the triangle inequality, we get from (4.31)

$$\|a_\varepsilon(\tau, \cdot) - \bar{a}(\tau, \cdot)\|_\infty \leq \|g_\varepsilon(\tau, \cdot)\|_\infty + C_\mathcal{L} \int_0^\tau du \|a_\varepsilon(u, \cdot) - \bar{a}(u, \cdot)\|_\infty. \quad (4.33)$$

Now Gronwall's inequality gives

$$\|a_\varepsilon(\tau, \cdot) - \bar{a}(\tau, \cdot)\|_\infty \leq e^{\tau C_\mathcal{L}} \sup_{0 < t < \tau} \|g_\varepsilon(t, \cdot)\|_\infty, \quad (4.34)$$

and we have

$$\sup_{0 < t < T} \|a_\varepsilon(\tau, \cdot) - \bar{a}(\tau, \cdot)\|_\infty \leq C' \sup_{0 < t < T} \|g_\varepsilon(t, \cdot)\|_\infty, \quad (4.35)$$

and $\sup_{0 < t < T} \|g_\varepsilon(\tau, \cdot)\|_\infty$ is estimated using the next lemma, proved in Appendix D.

LEMMA 4.1. *Suppose that $\mu(t, z)$ satisfies the mixing assumption (3.11) and similar for μ_z . Then for $0 < T' < T$*

$$\mathbb{P} \left[\sup_{\tau \in [0, \varepsilon^q T], s \in [-S, S]} \left| \int_{T'}^{T'+\tau} du [(\mathcal{L}_\varepsilon(u) - \bar{\mathcal{L}})\varphi](s) \right| > c\varepsilon^{\frac{\theta-\vartheta}{2}} \right] < C\varepsilon^{\vartheta+q},$$

for any deterministic continuous function $\varphi(s)$, arbitrary $q > 0$ and ϑ satisfying $\max[0, 2(\theta - 1)] < \vartheta < \theta$. Here θ is as in Lemma 3.1.

Let us introduce a uniform discretization of the time, in steps Δ , and write the residual (4.32) as

$$g_\varepsilon(\tau, s) = \sum_{j=0}^{\lfloor \tau/\Delta \rfloor - 1} \int_{j\Delta}^{(j+1)\Delta} du [(\mathcal{L}_\varepsilon(u) - \bar{\mathcal{L}})\bar{a}(u, \cdot)](s) + \int_{\Delta \lfloor \tau/\Delta \rfloor}^{\tau} du [(\mathcal{L}_\varepsilon(u) - \bar{\mathcal{L}})\bar{a}(u, \cdot)](s). \quad (4.36)$$

The last term is bounded by $C\Delta$ uniformly in τ and s , because both \mathcal{L}_ε and $\bar{\mathcal{L}}$ are bounded linear operators and $\bar{a}(\tau, s)$ is bounded. Moreover, (4.27) and (4.28) give for $j\Delta \leq u \leq (j+1)\Delta$ that

$$\|\mathcal{L}_\varepsilon(u)\bar{a}(u, \cdot) - \mathcal{L}_\varepsilon(u)\bar{a}(j\Delta, \cdot)\|_\infty \leq C_\mathcal{L} \|\bar{a}(u, \cdot) - \bar{a}(j\Delta, \cdot)\|_\infty \leq C(u - j\Delta),$$

where C is a constant that is independent of j , Δ and ε . A similar result holds for $\bar{\mathcal{L}}$, so we get

$$\begin{aligned} \|g_\varepsilon(\tau, \cdot)\|_\infty &\leq \sum_{j=0}^{\lfloor \tau/\Delta \rfloor - 1} \sup_{s \in [-S, S]} \left| \int_{j\Delta}^{(j+1)\Delta} du [(\mathcal{L}_\varepsilon(u) - \bar{\mathcal{L}})\bar{a}(j\Delta, \cdot)](s) \right| + \\ &C \sum_{j=0}^{\lfloor \tau/\Delta \rfloor - 1} \int_{j\Delta}^{(j+1)\Delta} du (u - j\Delta) + C'\Delta, \end{aligned} \quad (4.37)$$

with redefined ε, Δ independent constants C and C' . The second term is

$$\sum_{j=0}^{\lfloor \tau/\Delta \rfloor - 1} \int_{j\Delta}^{(j+1)\Delta} du (u - j\Delta) = \frac{1}{2} \sum_{j=0}^{\lfloor \tau/\Delta \rfloor - 1} \Delta^2 \leq \tau\Delta/2,$$

and thus together with the last term vanishes in the limit $\Delta \rightarrow 0$. In view of Lemma 4.1 we get by choosing $\Delta = \varepsilon^q$ and $q = (\theta - \vartheta)/4$ that

$$\mathbb{P} \left[\sup_{\tau \leq T} \|g_\varepsilon(\tau, \cdot)\|_\infty > c\varepsilon^{\frac{\theta-\vartheta}{4}} \right] < C\varepsilon^\vartheta,$$

which gives the result.

5. Summary. We have considered wave propagation in one dimensional random media in the situation when the wave speed varies randomly in both space and time. We derived a mathematical framework involving a random integral equation that can be used to characterize the pulse in this case. The main result is that the pulse stabilization phenomenon known in time independent random media extends to the time variable case. Pulse stabilization means that the amplitude of the wave transmitted from the source to range z has a deterministic form, when observed in a time window centered at $\tau + \delta\tau$, and of width comparable to the support of the initial pulse. Here τ is the deterministic travel time to range z , and $\delta\tau$ is a small random correction, on the scale of the support of the initial pulse. It is well known that in time independent media the transmitted pulse is faded and broadened, because scattering in the medium produces delays and also transfers energy to the incoherent waves. We assume random temporal changes in the medium, modeled by temporal fluctuations of the wave speed. We call the time fluctuations slow when the life span of a spatial realization of the random wave speed is longer than the time of traversal of a correlation length and the pulse width. The fluctuations are rapid when these time scales are similar. The results show that the slow fluctuations have to leading order no effect on the wave front. The rapid fluctuations affect both the random arrival time correction $\delta\tau$ and the deterministic pulse deformation. In particular, there is a trade-off between frequency dependent attenuation due to scattering in the medium, and energy gain induced by the temporal fluctuations.

The analysis is based on a stochastic time domain integral equation satisfied by the wave front, and the pulse stabilization follows from stochastic averaging. Furthermore, the energy gain induced by the temporal fluctuations is explained by drawing an analogy to the random harmonic oscillator problem.

Acknowledgments. The work of L. Borcea was partially supported by the ONR Grant N00014-14-1-0077. Support from the AFOSR Grant FA9550-12-1-0117 is also gratefully acknowledged. The work of K. Sølna was partially supported by AFSOR grant # FA 9550-11-1-0176.

Appendix A. Derivation of the first order system for the right and left going waves. Let us take the z derivative in the first equation in (3.1) to obtain

$$\frac{\partial A_\varepsilon}{\partial z} = \zeta_\varepsilon^{-1/2} \frac{\partial p_\varepsilon}{\partial z} - \frac{p_\varepsilon}{2\zeta_\varepsilon^{3/2}} \frac{\partial \zeta_\varepsilon}{\partial z} + \zeta_\varepsilon^{1/2} \frac{\partial u_\varepsilon}{\partial z} + \frac{u_\varepsilon}{2\zeta_\varepsilon^{1/2}} \frac{\partial \zeta_\varepsilon}{\partial z}. \quad (\text{A.1})$$

Since p_ε and u_ε satisfy (2.1), we have for $z \neq 0$

$$\frac{\partial A_\varepsilon}{\partial z} = -\rho \zeta_\varepsilon^{-1/2} \frac{\partial u_\varepsilon}{\partial t} - \frac{p_\varepsilon}{2\zeta_\varepsilon^{3/2}} \frac{\partial \zeta_\varepsilon}{\partial z} - \zeta_\varepsilon^{1/2} K_\varepsilon^{-1} \frac{\partial p_\varepsilon}{\partial t} + \frac{u_\varepsilon}{2\zeta_\varepsilon^{1/2}} \frac{\partial \zeta_\varepsilon}{\partial z}. \quad (\text{A.2})$$

Equations (3.1) give that $p_\varepsilon = \zeta_\varepsilon^{1/2}(A_\varepsilon - B_\varepsilon)/2$ and $u_\varepsilon = \zeta_\varepsilon^{-1/2}(A_\varepsilon + B_\varepsilon)/2$, so

$$\begin{aligned} \frac{\partial p_\varepsilon}{\partial t} &= \frac{\zeta_\varepsilon^{1/2}}{2} \left(\frac{\partial A_\varepsilon}{\partial t} - \frac{\partial B_\varepsilon}{\partial t} \right) + \frac{1}{4\zeta_\varepsilon^{1/2}} \frac{\partial \zeta_\varepsilon}{\partial t} (A_\varepsilon - B_\varepsilon) \\ \frac{\partial u_\varepsilon}{\partial t} &= \frac{\zeta_\varepsilon^{-1/2}}{2} \left(\frac{\partial A_\varepsilon}{\partial t} + \frac{\partial B_\varepsilon}{\partial t} \right) - \frac{1}{4\zeta_\varepsilon^{3/2}} \frac{\partial \zeta_\varepsilon}{\partial t} (A_\varepsilon + B_\varepsilon). \end{aligned} \quad (\text{A.3})$$

Substituting in (A.2) and using that $\rho/\zeta_\varepsilon = 1/c_\varepsilon$, we obtain

$$\frac{\partial A_\varepsilon}{\partial z} = -\frac{1}{c_\varepsilon} \frac{\partial A_\varepsilon}{\partial t} + \left(\frac{1}{2\zeta_\varepsilon} \frac{\partial \zeta_\varepsilon}{\partial z} + \frac{1}{2c_\varepsilon \zeta_\varepsilon} \frac{\partial \zeta_\varepsilon}{\partial t} \right) B_\varepsilon. \quad (\text{A.4})$$

Equation (4.1) follows with

$$M_\varepsilon(t, z) = \frac{\varepsilon}{2\zeta_\varepsilon} \frac{\partial \zeta_\varepsilon}{\partial z}, \quad N_\varepsilon(t, z) = \frac{\varepsilon}{2c_\varepsilon \zeta_\varepsilon} \frac{\partial \zeta_\varepsilon}{\partial t}. \quad (\text{A.5})$$

Equation (4.2) follows similarly. \square

Appendix B. Mixing assumption. For completeness we show here that the mixing conditions (3.10) and (3.11) are implied by a ϕ -mixing assumption, see also [14, 16]. To state the assumption, let (Ω, \mathcal{F}, P) be a probability space and denote by $\mathcal{F}_{z_1}^{z_2}$ a family of σ -algebras contained in \mathcal{F} , so that $\mathcal{F}_{z_2}^{z_3} \subset \mathcal{F}_{z_1}^{z_4}$, $0 \leq z_1 \leq z_2 \leq z_3 \leq z_4 \leq \infty$. The ϕ -mixing assumption is

$$\sup_{z \geq 0} \sup_{U \in \mathcal{F}_{z+\Delta z}^\infty, V \in \mathcal{F}_0^z} |P(U|V) - P(U)| \leq (\Delta z)^{-d}, \quad \Delta z > 0, \quad (\text{B.1})$$

with $d > 2$. We assume that conditional probabilities relative to \mathcal{F}_0^z , $0 \leq z \leq \infty$, have a regular version so that with probability one we have the representation

$$\mathbb{E}[\cdot | \mathcal{F}_0^z](\omega') = \int_{\Omega} \cdot \mathbb{P}_z(d\omega | \omega').$$

Assume that X is a real valued, bounded in absolute value by c and \mathcal{F}_0^z measurable random variable. Then

$$|\mathbb{E}[X | \mathcal{F}_0^z](\omega') - \mathbb{E}[X]| = \left| \int_{\Omega} X(\omega) [\mathbb{P}_z(d\omega | \omega') - \mathbb{P}(d\omega)] \right| \quad (\text{B.2})$$

$$= \left| \int_{\Omega} X(\omega) v_z(d\omega | \omega') \right|. \quad (\text{B.3})$$

Here, $v_z(U | \omega') = \mathbb{P}_z(U | \omega') - \mathbb{P}(U)$ is a signed measure and, from (B.1) and the Hahn decomposition theorem, its variation $|v_z|(U | \omega')$ satisfies

$$\sup_{U \in \mathcal{F}_{z+\Delta z}^\infty} |v_z|(U | \omega') \leq 2(\Delta z)^{-d}.$$

Then we consequently have

$$|\mathbb{E}[X | \mathcal{F}_0^z](\omega') - \mathbb{E}[X]| \leq c2(\Delta z)^{-d}. \quad (\text{B.4})$$

The bounds (3.10), respectively (3.11), follow accordingly by considering the filtration generated by $\mu(\cdot, z)$ and the random variable $\mu(t, z)$, respectively $\mu(t_1, z_1)\mu(t_2, z_2)$ (moreover their derivatives).

Appendix C. Proof of Lemma 3.1. We prove the lemma in regime 1, where $\alpha < 1$ and $\beta = 2$. The results extend to regime 2 using a change of variables, as explained in section C.4. We begin in section C.1 with the estimation of the residual

$$q_\varepsilon(z, s) = \mathcal{W}_\varepsilon\left(z/c_o, s\right) - W_\varepsilon(z/c_o), \quad (\text{C.1})$$

where W_ε is defined in (3.12). The estimation of $\partial_s \mathcal{W}_\varepsilon(z/c_o, s)$ is in section C.3.

We use henceforth the notation c and C for constants that are independent of ε .

C.1. Proof of estimate (3.13). Let us recall definitions (3.6) and (3.12) to write the residual (C.1) as

$$\begin{aligned} q_\varepsilon(z, s) &= \varepsilon^{1-\alpha} \int_0^z du \nu_\varepsilon(u, s) [s + W_\varepsilon((u - \varepsilon^p)/c_o) + q_\varepsilon(u - \varepsilon^p, s)] + \\ &\quad \varepsilon^{1-\alpha} \int_0^z du \tilde{\nu}_\varepsilon(u, s) [W_\varepsilon(u/c_o) - W_\varepsilon((u - \varepsilon^p)/c_o) + q_\varepsilon(u, s) - q_\varepsilon(u - \varepsilon^p, s)], \end{aligned} \quad (\text{C.2})$$

for p satisfying

$$\alpha < p < 1 + \alpha/2. \quad (\text{C.3})$$

By the mean value theorem

$$\begin{aligned} \mu\left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}\xi_\varepsilon(u, s), \frac{u}{\varepsilon^2}\right) - \mu\left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}\eta_\varepsilon(u, s), \frac{u}{\varepsilon^2}\right) &= \varepsilon^{2-\alpha} \times \\ &\quad [\xi_\varepsilon(u, s) - \eta_\varepsilon(u, s)] \mu_t\left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}\zeta_\varepsilon(u, s), \frac{u}{\varepsilon^2}\right), \end{aligned}$$

with $\zeta_\varepsilon(u, s)$ between $\xi_\varepsilon(u, s)$ and $\eta_\varepsilon(u, s)$. In the first term in (C.2) we have

$$\nu_\varepsilon(u, s) = c_o^{-1} \mu_t\left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha}\zeta_\varepsilon(u, s), \frac{u}{\varepsilon^2}\right), \quad (\text{C.4})$$

and $\zeta_\varepsilon(u, s)$ is between $\xi_\varepsilon(u, s) = s + W_\varepsilon((u - \varepsilon^p)/c_o) + q_\varepsilon(u - \varepsilon^p, s)$ and $\eta_\varepsilon = 0$. The expression of $\tilde{\nu}_\varepsilon$ in the second term of (C.2) is similar, except that $\zeta_\varepsilon(u, s)$ is between $\xi_\varepsilon(u, s) = s + W_\varepsilon(u/c_o) + q_\varepsilon(u, s)$ and $\eta_\varepsilon(u, s) = s + W_\varepsilon((u - \varepsilon^p)/c_o) + q_\varepsilon(u - \varepsilon^p, s)$. Using that $q_\varepsilon(u, s)$ is supported for positive u , we write equation (C.2) as

$$q_\varepsilon(z, s) = \varepsilon^{1-\alpha} \int_0^z du \kappa_\varepsilon(u, z, s) q_\varepsilon(u, s) + \mathcal{T}_{\varepsilon,1}(z, s) + \mathcal{T}_{\varepsilon,2}(z, s), \quad (\text{C.5})$$

with the notation

$$\begin{aligned} \mathcal{T}_{\varepsilon,1}(z, s) &= \varepsilon^{1-\alpha} \int_0^z du [\tilde{\nu}_\varepsilon(u, s) - \nu_\varepsilon(u, s)] [W_\varepsilon(u/c_o) - W_\varepsilon((u - \varepsilon^p)/c_o)], \\ \mathcal{T}_{\varepsilon,2}(z, s) &= \varepsilon^{1-\alpha} \int_0^z du \nu_\varepsilon(u, s) [s + W_\varepsilon(u/c_o)], \end{aligned}$$

and integral kernel

$$\kappa_\varepsilon(u, z, s) = \tilde{\nu}_\varepsilon(u, s) + [\nu_\varepsilon(u + \varepsilon^p, s) - \tilde{\nu}_\varepsilon(u + \varepsilon^p, s)] 1_{[0, z - \varepsilon^p]}(u),$$

where $1_{[m, M]}(u)$ is the indicator function of the interval $[m, M]$. We can bound q_ε using Gronwall's inequality, once we estimate all the terms in (C.5).

Definition (3.12) and the boundedness of μ and μ_t imply that

$$\sup_{0 \leq u \leq z \leq L} \varepsilon^{1-\alpha} |\kappa_\varepsilon(u, z, s)| \leq C \varepsilon^{1-\alpha}, \quad \sup_{z \in [0, L]} |\mathcal{T}_{\varepsilon,1}(z, s)| \leq C \varepsilon^{p-\alpha},$$

for all $s \in [0, S]$. The estimation of $\mathcal{T}_{\varepsilon,2}$ is more involved, and the result proved in section C.2 is

$$\mathbb{P}\left(\sup_{z \in [0, L]} |\mathcal{T}_{\varepsilon,2}(z, s)| \geq c \varepsilon^\vartheta\right) \leq C \varepsilon^{2-\alpha-\vartheta}, \quad (\text{C.6})$$

with ϑ satisfying $0 < \vartheta \leq p - \alpha < 2 - \alpha$. Then (C.5), the assumption $\alpha \leq 1$ and Gronwall's inequality give

$$\mathbb{P}\left(\sup_{z \in [0, L]} |q_\varepsilon(z, s)| \geq c\varepsilon^\vartheta\right) \leq C\varepsilon^{2-\alpha-\vartheta}, \quad \forall s \in [0, S]. \quad (\text{C.7})$$

To obtain a uniform estimate in s we note that since the characteristics cannot intersect, the function $s + q_\varepsilon(z, s)$ is strictly monotonically increasing in s , for all $z \in [0, L]$. More explicitly,

$$q_\varepsilon(z, s_1) - (s_2 - s_1) < q_\varepsilon(z, s_2) < q_\varepsilon(z, s_3) + (s_3 - s_2), \quad (\text{C.8})$$

for all $0 \leq s_1 < s_2 < s_3 \leq S$. Let us discretize the interval $[0, S]$ in steps equal to $\varepsilon^\vartheta S$, with discretization points s_j , for $0 \leq j \leq S\varepsilon^{-\vartheta}$. For each s_j denote by E_j the event

$$E_j = \{\omega \mid \sup_{0 \leq z \leq L} |q_\varepsilon(z, s_j)| \geq c\varepsilon^\vartheta\},$$

with $\mathbb{P}(E_j) \leq C\varepsilon^{2-\alpha-\vartheta}$ following from (C.7). The union of these events has probability

$$\mathbb{P}\left(\cup E_j\right) \leq \sum_{0 \leq j \leq S\varepsilon^{-\vartheta}} \mathbb{P}(E_j) \leq C\varepsilon^{2-\alpha-2\vartheta},$$

where the assumption (C.3) on p guarantees that $2\vartheta \leq 2(p - \alpha) < 2(2 - \alpha)$. Finally, the monotonicity relation (C.8) gives that

$$\sup_{z \in [0, L], s \in [0, S]} |q_\varepsilon(z, s)| < c\varepsilon^\vartheta,$$

on the complement of $\cup E_j$, with probability $1 - O(\varepsilon^{2-\alpha-2\vartheta})$ and redefined constant c . This is the statement (3.13) in Lemma 3.1, with $\theta = 2 - \alpha - 2\vartheta$ and $L = c_o T$.

C.2. Proof of estimate (C.6). Let us write $\mathcal{T}_{\varepsilon, 2}$ as the sum of two parts

$$\mathcal{T}_{\varepsilon, 2}(z, s) = s\mathcal{J}(z, s) + \varepsilon^{\alpha-2}\mathcal{I}(z, s), \quad (\text{C.9})$$

where

$$\begin{aligned} \mathcal{J}(z, s) &= \varepsilon^{1-\alpha} \int_0^z du \nu_\varepsilon(u, s), \\ \mathcal{I}(z, s) &= \varepsilon^{1-\alpha} \int_0^z du \nu_\varepsilon(u, s) \varepsilon^{1-\alpha} \int_0^u dv \mu\left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right). \end{aligned}$$

We estimate \mathcal{J} in section C.2.1 and \mathcal{I} in section C.2.2.

C.2.1. Estimate of $\mathcal{J}(z, s)$. We begin with proving the basic result

$$\left| \varepsilon^{1-\alpha} \mathbb{E}\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right] \right| \leq C\varepsilon^{3-\alpha}, \quad (\text{C.10})$$

for all $s \in [0, S]$ and integers m satisfying $1 \leq m \leq \lfloor L/\varepsilon \rfloor$. We write**

$$\mathbb{E}\left[\int_{(m-1)\varepsilon}^{(m-1)\varepsilon+\varepsilon^2} du + \int_{(m-1)\varepsilon+\varepsilon^2}^{(m-1)\varepsilon+\varepsilon^p} du + \int_{(m-1)\varepsilon+\varepsilon^p}^{m\varepsilon} du \nu_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right] = T_1 + T_2 + T_3,$$

**Here we split the integral in three parts to account for all p satisfying (C.3). If p were ≤ 1 , then the part T_3 would not be necessary, because $(m-1)\varepsilon + \varepsilon^p$ would exceed $m\varepsilon$.

with ν_ε defined in (C.4), for $\zeta_\varepsilon(u, s)$ between 0 and $s + W_\varepsilon(u - \varepsilon^p) + q_\varepsilon(u - \varepsilon^p, s)$. Since ν_ε is bounded, $|T_1| \leq C\varepsilon^2$. For T_2 we can use the mixing assumption (3.10), because $\zeta_\varepsilon(u, s)$ is $\mathcal{F}_{\frac{m-1}{\varepsilon}}$ measurable for all $u \leq (m-1)\varepsilon + \varepsilon^p$, and obtain

$$|T_2| \leq C \int_{(m-1)\varepsilon + \varepsilon^2}^{(m-1)\varepsilon + \varepsilon^p} du \left(\frac{u}{\varepsilon^2} - \frac{m-1}{\varepsilon} \right)^{-d} = \frac{C}{d-1} \varepsilon^2 \left(1 - \varepsilon^{(d-1)(2-p)} \right).$$

Our assumptions on p and d give that $(d-1)(2-p) > 0$, so \mathcal{T}_2 is $O(\varepsilon^2)$. For T_3 we have $u - \varepsilon^p \geq (m-1)\varepsilon$, so definition (C.4), the tower property of the expectation and the mixing assumption (3.10) give

$$\begin{aligned} |T_3| &= c_o^{-1} \left| \int_{(m-1)\varepsilon + \varepsilon^p}^{m\varepsilon} du \mathbb{E} \left[\mathbb{E} \left[\mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \zeta_\varepsilon(u - \varepsilon^p, s), \frac{u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{u-\varepsilon^p}{\varepsilon^2}} \right] \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right| \\ &\leq C\varepsilon \left(\frac{\varepsilon^p}{\varepsilon^2} \right)^{-d} \ll C\varepsilon^2. \end{aligned}$$

Here we used that $d > (2-p)^{-1}$ by (C.3), and $d > 2$. Estimate (C.10) follows.

Now let us define

$$Y_m = \mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s) - \mathbb{E} \left[\mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right],$$

for^{††} $1 \leq m \leq L/\varepsilon$, and

$$X_m = \sum_{j=1}^m Y_j,$$

which is easily verified to be a martingale, using the tower property of the conditional expectation. Then,

$$\mathcal{J}(m\varepsilon, s) = X_m + \sum_{r=1}^m \mathbb{E} \left[\mathcal{J}(r\varepsilon, s) - \mathcal{J}((r-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{r-1}{\varepsilon}} \right], \quad (\text{C.11})$$

with terms in the sum bounded like in the left hand side in (C.10), and X_m bounded by Doob's Submartingale Inequality

$$\mathbb{P} \left(\sup_{m \leq L/\varepsilon} |X_m| \geq c\varepsilon^{(2-\alpha+\vartheta)/2} \right) \leq \frac{\varepsilon^{-(2-\alpha+\vartheta)}}{c^2} \mathbb{E} \left[X_{L/\varepsilon}^2 \right] = \frac{\varepsilon^{-(2-\alpha+\vartheta)}}{c^2} \sum_{j=1}^{L/\varepsilon} \mathbb{E} [Y_j^2].$$

In the last equation we used the tower property of the conditional expectation,

$$\mathbb{E} \left[\mathbb{E} \left[\mathcal{J}(r\varepsilon, s) - \mathcal{J}((r-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{r-1}{\varepsilon}} \right] \middle| \mathcal{F}_{\frac{j}{\varepsilon}} \right] = \mathbb{E} \left[\mathcal{J}(r\varepsilon, s) - \mathcal{J}((r-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{j}{\varepsilon}} \right]$$

for $1 \leq j < r \leq L/\varepsilon$, to get that $\mathbb{E}[Y_j Y_r] = \mathbb{E}[Y_j \mathbb{E}[Y_r | \mathcal{F}_{\frac{j}{\varepsilon}}]] = 0$. By the definition of $\mathcal{J}(z, s)$ and the boundedness of μ_t

$$|\mathcal{J}(z, s) - \mathcal{J}((m-1)\varepsilon, s)| \leq C\varepsilon^{2-\alpha}, \quad \forall z \in ((m-1)\varepsilon, m\varepsilon), \quad s \in [0, S],$$

^{††}For convenience we treat henceforth L/ε as an integer. If it is not, there is an additional error term which does not make any difference because it tends to zero faster than ε .

so using (C.11) terms in the sum estimated by (C.10) we get

$$\sup_{z \in [0, L]} |\mathcal{J}(z, s)| \leq C\varepsilon^{2-\alpha} + \sup_{m \leq L/\varepsilon} |X_m|. \quad (\text{C.12})$$

It remains to bound the martingale X_m .

The definition of Y_m and the inequality $(a+b)^2 \leq 2(a^2+b^2)$ give

$$\begin{aligned} \mathbb{E}[Y_m^2] &\leq 2\mathbb{E}\left[|\mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s)|^2\right] + \\ &\quad 2\mathbb{E}\left[\left|\mathbb{E}\left[\mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right]\right|^2\right], \end{aligned}$$

and using Jensen's inequality

$$\left|\mathbb{E}\left[\mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right]\right|^2 \leq \mathbb{E}\left[|\mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s)|^2 \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right],$$

and the tower property of the expectation, we get

$$\begin{aligned} \mathbb{E}[Y_m^2] &\leq 4\mathbb{E}\left[|\mathcal{J}(m\varepsilon, s) - \mathcal{J}((m-1)\varepsilon, s)|^2\right] \\ &= 4\varepsilon^{2(1-\alpha)}\mathbb{E}\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \int_{(m-1)\varepsilon}^{m\varepsilon} dv \nu_\varepsilon(v, s)\right] \\ &= 8\varepsilon^{2(1-\alpha)}\mathbb{E}\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \mathbb{E}\left[\int_u^{m\varepsilon} dv \nu_\varepsilon(v, s) \middle| \mathcal{F}_{\frac{u}{\varepsilon^2}}\right]\right]. \end{aligned}$$

In the last equation we wrote the integral over the square $(u, v) \in [(m-1)\varepsilon, m\varepsilon] \times [(m-1)\varepsilon, m\varepsilon]$ as twice the integral over its half, the triangle $u \in [(m-1)\varepsilon, m\varepsilon]$ and $v \in [u, m\varepsilon]$, and used the tower property of the expectation. The inner expectation is estimated as in (C.10) and the result is

$$\mathbb{E}[Y_m^2] \leq C\varepsilon^{1+2(2-\alpha)}.$$

Substituting in Doob's Submartingale Inequality and then in (C.12) we get

$$\mathbb{P}\left(\sup_{z \in [0, L]} |\mathcal{J}(z, s)| \geq c\varepsilon^{(2-\alpha+\vartheta)/2}\right) \leq C\varepsilon^{2-\alpha-\vartheta}, \quad \forall s \in [0, S], \quad (\text{C.13})$$

with redefined constants c and C .

C.2.2. Estimate of $\mathcal{I}(z, s)$. We can rewrite the expression of $\mathcal{I}(z, s)$ as

$$\mathcal{I}(z, s) = \varepsilon^{1-\alpha} \int_0^z du \nu_\varepsilon(u, s) J(u), \quad (\text{C.14})$$

where

$$J(u) = \varepsilon^{1-\alpha} \int_0^u dv \mu\left(\frac{v/c_0}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right) \quad (\text{C.15})$$

can be estimated the same way as $\mathcal{J}(z, s)$. We write directly the result

$$\mathbb{P}\left(\sup_{0 \leq z \leq L} |J(z)| \geq c\varepsilon^{(2-\alpha+\vartheta)/2}\right) \leq C\varepsilon^{2-\alpha-\vartheta}. \quad (\text{C.16})$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[J^2(z)\right] &= \varepsilon^{2(1-\alpha)} \iint_0^z du dv \mathbb{E}\left[\mu\left(\frac{u/c_o}{\varepsilon^\alpha}, \frac{u}{\varepsilon^2}\right)\mu\left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right)\right] \\ &= \varepsilon^{2(1-\alpha)} \iint_0^z du dv \Phi\left(\frac{u-v}{c_o\varepsilon^\alpha}, \frac{u-v}{\varepsilon^2}\right),\end{aligned}$$

and recalling that the autocorrelation Φ is integrable,

$$\sup_{0 \leq z \leq L} \mathbb{E}\left[J^2(z)\right] \leq C\varepsilon^{2(2-\alpha)} \leq C\varepsilon^{2-\alpha+\vartheta}. \quad (\text{C.17})$$

Now, we obtain from (C.14) that

$$\begin{aligned}\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) &= J((m-1)\varepsilon) \varepsilon^{1-\alpha} \int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) + \\ &\quad \varepsilon^{1-\alpha} \int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) [J(u) - J((m-1)\varepsilon)],\end{aligned}$$

and taking the conditional expectation

$$\begin{aligned}\mathbb{E}\left[\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right] &= J((m-1)\varepsilon) \mathbb{E}\left[\varepsilon^{1-\alpha} \int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right] + \\ &\quad \varepsilon^{2(1-\alpha)} \mathbb{E}\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \int_{(m-1)\varepsilon}^u dv \mu\left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right].\end{aligned}$$

We change the order of integration in the last term and condition at v to get

$$\begin{aligned}\mathbb{E}\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \int_{(m-1)\varepsilon}^u dv \mu\left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right] &= \\ \mathbb{E}\left[\int_{(m-1)\varepsilon}^{m\varepsilon} dv \mu\left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right) \mathbb{E}\left[\int_v^{m\varepsilon} du \nu_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{v}{\varepsilon^2}}\right] \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right].\end{aligned}$$

Then estimates (C.10) and (C.16) give

$$\begin{aligned}\left|\mathbb{E}\left[\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}}\right]\right| 1_E &\leq C' \varepsilon^{3-\alpha+(2-\alpha+\vartheta)/2} [1 + O(\varepsilon^{(2-\alpha-\vartheta)/2})] \\ &\leq C\varepsilon^{1+(2-\alpha+\vartheta)},\end{aligned} \quad (\text{C.18})$$

where we renamed the constant in the last inequality, used that $\vartheta < 2 - \alpha$ and μ is bounded, and let E be the event with probability $\mathbb{P}(E) \geq 1 - O(\varepsilon^{2-\alpha-\vartheta})$ so that $\sup_{z \leq L} |J(z)| 1_E \leq \varepsilon^{(2-\alpha+\vartheta)/2}$.

We calculate next the second moment of the increments of $\mathcal{I}(z, s)$,

$$\begin{aligned}\mathbb{E}\left[\left[\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s)\right]^2\right] &\leq 2\varepsilon^{2(1-\alpha)} \mathbb{E}\left[J^2((m-1)\varepsilon) \left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s)\right]^2\right] + \\ &\quad 2\varepsilon^{4(1-\alpha)} \mathbb{E}\left[\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \int_{(m-1)\varepsilon}^u dv \mu\left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2}\right)\right]^2\right] = T_1 + T_2.\end{aligned}$$

To estimate the first term we write the square of the integral of ν_ε as

$$\left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s)\right]^2 = 2 \int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \int_u^{m\varepsilon} dv \nu_\varepsilon(v, s),$$

condition at u , and use (C.10) and (C.17) to get

$$\begin{aligned} T_1 &= 4\varepsilon^{2(1-\alpha)} \mathbb{E} \left[J^2((m-1)\varepsilon) \int_{(m-1)\varepsilon}^{m\varepsilon} du \nu_\varepsilon(u, s) \mathbb{E} \left[\int_u^{m\varepsilon} dv \nu_\varepsilon(v, s) \middle| \mathcal{F}_{\frac{u}{\varepsilon^2}} \right] \right] \\ &\leq C\varepsilon^{1+3(2-\alpha)+\vartheta}. \end{aligned}$$

The second term is

$$\begin{aligned} T_2 &= 2\varepsilon^{4(1-\alpha)} \mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} dv \mu \left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2} \right) \int_{(m-1)\varepsilon}^{m\varepsilon} dv' \mu \left(\frac{v'/c_o}{\varepsilon^\alpha}, \frac{v'}{\varepsilon^2} \right) \times \right. \\ &\quad \left. \int_v^{m\varepsilon} du \nu_\varepsilon(u, s) \int_{v'}^{m\varepsilon} du' \nu_\varepsilon(u', s) \right] = T_2^{(i)} + T_2^{(ii)} \end{aligned}$$

and we split it in two parts

$$\begin{aligned} T_2^{(i)} &= 2\varepsilon^{4(1-\alpha)} \mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} dv \mu \left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2} \right) \int_{(m-1)\varepsilon}^v dv' \mu \left(\frac{v'/c_o}{\varepsilon^\alpha}, \frac{v'}{\varepsilon^2} \right) \times \right. \\ &\quad \left. \left[\int_v^{m\varepsilon} du \nu_\varepsilon(u, s) \int_{v'}^v du' \nu_\varepsilon(u', s) + \int_v^{m\varepsilon} du \nu_\varepsilon(u, s) \int_v^{m\varepsilon} du' \nu_\varepsilon(u', s) \right] \right], \end{aligned}$$

and

$$\begin{aligned} T_2^{(ii)} &= 2\varepsilon^{4(1-\alpha)} \mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} dv \mu \left(\frac{v/c_o}{\varepsilon^\alpha}, \frac{v}{\varepsilon^2} \right) \int_v^{m\varepsilon} dv' \mu \left(\frac{v'/c_o}{\varepsilon^\alpha}, \frac{v'}{\varepsilon^2} \right) \times \right. \\ &\quad \left. \left[\int_{v'}^{m\varepsilon} du' \nu_\varepsilon(u', s) \int_v^{v'} du \nu_\varepsilon(u, s) + \int_{v'}^{m\varepsilon} du' \nu_\varepsilon(u', s) \int_{v'}^{m\varepsilon} du \nu_\varepsilon(u, s) \right] \right]. \end{aligned}$$

We condition at v in the first term of $T_2^{(i)}$, and use the estimate (C.10) for the u integral. The second term involves the integral

$$\int_v^{m\varepsilon} du \nu_\varepsilon(u, s) \int_v^{m\varepsilon} du' \nu_\varepsilon(u', s) = 2 \int_v^{m\varepsilon} du \nu_\varepsilon(u, s) \int_u^{m\varepsilon} du' \nu_\varepsilon(u', s)$$

and we can estimate it using (C.10), after conditioning at u . We obtain

$$T_2^{(i)} \leq C\varepsilon^{1+4(2-\alpha)},$$

and proceeding in the same manner we obtain a similar bound for $T_2^{(ii)}$. Gathering the results we have

$$\mathbb{E} \left[\left[\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) \right]^2 \right] \leq C\varepsilon^{1+3(2-\alpha)+\vartheta}, \quad (\text{C.19})$$

because $\vartheta < (2-\alpha)$.

Now define the martingale $X_m = \sum_{j=1}^m Y_m$, with

$$Y_m = \mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) - \mathbb{E} \left[\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right],$$

and get the bound

$$\begin{aligned} |\mathcal{I}(m\varepsilon, s)|_{1_E} &\leq |X_m|_{1_E} + \sum_{j=1}^m \left| \mathbb{E} \left[\mathcal{I}(m\varepsilon, s) - \mathcal{I}((m-1)\varepsilon, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right|_{1_E} \\ &\leq |X_m|_{1_E} + Cm\varepsilon\varepsilon^{2-\alpha+\vartheta} \end{aligned} \quad (\text{C.20})$$

where we used (C.18). The martingale is bounded with Doob's Submartingale Inequality

$$\mathbb{P}\left(\sup_{m \leq L/\varepsilon} |X_m| \geq c\varepsilon^{2-\alpha+\vartheta}\right) \leq \frac{\varepsilon^{-2(2-\alpha+\vartheta)}}{c^2} \mathbb{E}\left[X_{L/\varepsilon}^2\right] = \frac{\varepsilon^{-2(2-\alpha+\vartheta)}}{c^2} \sum_{j=1}^{L/\varepsilon} \mathbb{E}[Y_j^2] \leq C\varepsilon^{2-\alpha-\vartheta}.$$

The equality is implied by the tower property of the expectation, as in the previous section. The last bound is derived using Jensen's inequality and (C.19).

Finally, definition (C.14) and (C.16) give that for any $z \in ((m-1)\varepsilon, m\varepsilon)$ and $s \in [0, S]$,

$$|\mathcal{I}(z, s) - \mathcal{I}((m-1)\varepsilon, s)|_{1E} \leq \varepsilon^{2-\alpha} \sup_{u \in [0, L]} |J(u)|_{1E} \leq \varepsilon^{2-\alpha+(2-\alpha+\vartheta)/2} \leq \varepsilon^{2-\alpha+\vartheta}.$$

The estimate

$$\mathbb{P}\left(\sup_{0 \leq z \leq L} |\mathcal{I}(z, s)| \geq c\varepsilon^{2-\alpha+\vartheta}\right) \leq C\varepsilon^{2-\alpha-\vartheta} \quad (\text{C.21})$$

follows, and together with (C.13) and definition (C.9) it gives (C.6).

C.3. Proof of estimate (3.14). We introduce the notation

$$\mathcal{U}_\varepsilon(z, s) = \partial_s \mathcal{W}_\varepsilon(z/c_o, s), \quad (\text{C.22})$$

and take the derivative with respect to s in definition (3.6) to obtain

$$\mathcal{U}_\varepsilon(z, s) = \varepsilon^{1-\alpha} \int_0^z du \gamma_\varepsilon(u, s) \mathcal{U}_\varepsilon(u, s) + \varepsilon^{1-\alpha} \int_0^z du \gamma_\varepsilon(u, s), \quad (\text{C.23})$$

with

$$\gamma_\varepsilon(z, s) = c_o^{-1} \mu_t \left(\frac{z/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(z/c_o, s)), \frac{z}{\varepsilon^2} \right). \quad (\text{C.24})$$

The kernel in the first integral satisfies

$$\sup_{0 \leq u \leq L} \varepsilon^{1-\alpha} |\gamma_\varepsilon(u, s)| \leq C\varepsilon^{1-\alpha},$$

because μ_t is bounded. The second integral is estimated exactly like in section C.2.1, once we prove the equivalent of the basic estimate (C.10) for γ_ε .

Let us rewrite (C.24) as

$$\gamma_\varepsilon(u, s) = c_o^{-1} \left[\mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right) + \mathcal{E}_\varepsilon(u, s) \right],$$

with p as in (C.3) and error

$$\begin{aligned} \mathcal{E}_\varepsilon(u, s) = & \mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(u/c_o, s)), \frac{u}{\varepsilon^2} \right) - \\ & \mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right). \end{aligned}$$

We have

$$\varepsilon^{1-\alpha} \mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \gamma_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] = c_o^{-1} \varepsilon^{1-\alpha} (T_1 + T_2 + T_3), \quad (\text{C.25})$$

where

$$\begin{aligned} T_1 &= \mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \mathcal{E}_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right], \\ T_2 &= \mathbb{E} \left[\int_{(m-1)\varepsilon}^{(m-1)\varepsilon + \varepsilon^2} du \mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right], \\ T_3 &= \mathbb{E} \left[\int_{(m-1)\varepsilon + \varepsilon^2}^{m\varepsilon} du \mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right]. \end{aligned}$$

Definition (C.22), the mean value theorem, the boundedness of μ_{tt} and the estimate

$$|\mathcal{W}_\varepsilon(u/c_o, s) - \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)| \leq C\varepsilon^{-1+p},$$

give that $|T_1| \leq C\varepsilon^{2+p-\alpha} \leq C\varepsilon^2$. The last inequality is because $p > \alpha$. The estimate $|T_2| \leq C\varepsilon^2$ follows from the boundedness of μ_t , and $T_3 = T_3^{(i)} + T_3^{(ii)}$, where

$$\begin{aligned} T_3^{(i)} &= \mathbb{E} \left[\int_{(m-1)\varepsilon + \varepsilon^2}^{(m-1)\varepsilon + \varepsilon^p} du \mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right], \\ T_3^{(ii)} &= \mathbb{E} \left[\int_{(m-1)\varepsilon + \varepsilon^p}^{m\varepsilon} du \mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right]. \end{aligned}$$

In $T_3^{(i)}$ we note that $\mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)$ is $\mathcal{F}_{\frac{m-1}{\varepsilon}}$ measurable, and use the mixing assumption (3.10)

$$|T_3^{(i)}| \leq C \int_{(m-1)\varepsilon + \varepsilon^2}^{(m-1)\varepsilon + \varepsilon^p} du \left(\frac{u}{\varepsilon^2} - \frac{m-1}{\varepsilon} \right)^{-d} = \frac{C\varepsilon^2}{d-1} [1 - \varepsilon^{(d-1)(2-p)}] = O(\varepsilon^2).$$

In $T_3^{(ii)}$ we have $u - \varepsilon^p \geq (m-1)\varepsilon$, so we use the tower property of the expectation and the mixing assumption (3.10) to obtain

$$\begin{aligned} |T_3^{(ii)}| &= \left| \int_{(m-1)\varepsilon + \varepsilon^p}^{m\varepsilon} du \mathbb{E} \left[\mathbb{E} \left[\mu_t \left(\frac{u/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon((u - \varepsilon^p)/c_o, s)), \frac{u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{u-\varepsilon^p}{\varepsilon^2}} \right] \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right| \\ &\leq C\varepsilon \left(\frac{\varepsilon^p}{\varepsilon^2} \right)^{-d} \ll C\varepsilon^2. \end{aligned}$$

The last inequality is because $d > (2-p)^{-1}$. Gathering the results we get the equivalent of (C.10)

$$\left| \varepsilon^{1-\alpha} \mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} du \gamma_\varepsilon(u, s) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right| \leq C\varepsilon^{3-\alpha},$$

and derive as in section C.2.1 that

$$\mathbb{P} \left(\sup_{z \in [0, L]} \left| \varepsilon^{1-\alpha} \int_0^z du \gamma_\varepsilon(u, s) \right| \geq c\varepsilon^{(2-\alpha+\vartheta)/2} \right) \leq C\varepsilon^{2-\alpha-\vartheta}.$$

By assumption $\vartheta < 2 - \alpha$, so we conclude from (C.23), the assumption $\alpha \leq 1$ and Gronwall's inequality that

$$\mathbb{P}\left(\sup_{z \in [0, L]} |\mathcal{U}_\varepsilon(z, s)| \geq c\varepsilon^\vartheta\right) \leq C\varepsilon^{2-\alpha-\vartheta}, \quad \forall s \in [0, S]. \quad (\text{C.26})$$

To make the result uniform in s , we now estimate the variation of \mathcal{U}_ε , denoted by

$$\Delta\mathcal{U}_\varepsilon(z, s, \Delta s) = \mathcal{U}_\varepsilon(z, s + \Delta s) - \mathcal{U}_\varepsilon(z, s),$$

for all Δs satisfying $0 \leq \Delta s \leq \varepsilon^\vartheta S$. Equation (C.23) gives

$$\begin{aligned} \Delta\mathcal{U}_\varepsilon(z, s, \Delta s) &= \varepsilon^{1-\alpha} \int_0^z du \Delta\gamma_\varepsilon(u, s, \Delta s) [1 + \mathcal{U}_\varepsilon(u, s)] + \\ &\quad \varepsilon^{1-\alpha} \int_0^z du \gamma_\varepsilon(u, s + \Delta s) \Delta\mathcal{U}_\varepsilon(u, s, \Delta s), \end{aligned} \quad (\text{C.27})$$

where we introduced the notation $\Delta\gamma_\varepsilon(z, s, \Delta s) = \gamma_\varepsilon(z, s + \Delta s) - \gamma_\varepsilon(z, s)$. The kernel multiplying $\Delta\mathcal{U}_\varepsilon$ in the last integral in (C.27) is uniformly bounded by

$$\sup_{u \in [0, L], \Delta s \in [0, \varepsilon^\vartheta S]} \left| \varepsilon^{1-\alpha} \gamma_\varepsilon(u, s + \Delta s) \right| \leq C\varepsilon^{1-\alpha},$$

because μ_t and therefore γ_ε are bounded. Moreover, using definition (C.24) and the mean value theorem in the definition of $\Delta\gamma_\varepsilon$ we get

$$\begin{aligned} \Delta\gamma_\varepsilon(z, s, \Delta s) &= c_o^{-1} \varepsilon^{2-\alpha} \mu_{tt} \left(\frac{z/c_o}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \zeta_\varepsilon(z, s, \Delta s), \frac{z}{\varepsilon^2} \right) \times \\ &\quad \left[\Delta s + \mathcal{W}_\varepsilon(z/c_o, s + \Delta s) - \mathcal{W}_\varepsilon(z/c_o, s) \right], \end{aligned}$$

with $\zeta_\varepsilon(z, s, \Delta s)$ somewhere between $s + \mathcal{W}_\varepsilon(z/c_o, s)$ and $s + \Delta s + \mathcal{W}_\varepsilon(z/c_o, s + \Delta s)$. By assumption μ_{tt} is uniformly bounded. To bound the second factor, we recall the monotonicity relation (C.8), and obtain that

$$0 \leq \Delta s + \mathcal{W}_\varepsilon(z/c_o, s + \Delta s) - \mathcal{W}_\varepsilon(z/c_o, s) \leq \varepsilon^\vartheta S + \mathcal{W}_\varepsilon(z/c_o, s + \varepsilon^\vartheta S) - \mathcal{W}_\varepsilon(z/c_o, s),$$

with right hand side estimated in (C.7) for any given s and $s + \varepsilon^\vartheta S$. Consequently,

$$\mathbb{P}\left(\sup_{z \in [0, L], \Delta s \in [0, \varepsilon^\vartheta S]} |\Delta\gamma_\varepsilon(z, s, \Delta s)| \geq c\varepsilon^{2-\alpha+\vartheta}\right) \leq C\varepsilon^{2-\alpha-\vartheta},$$

and recalling the bound (C.26), we get

$$\mathbb{P}\left(\sup_{z \in [0, L], \Delta s \in [0, \varepsilon^\vartheta S]} \left| \varepsilon^{1-\alpha} \int_0^z du \Delta\gamma_\varepsilon(u, s, \Delta s) [1 + \mathcal{U}_\varepsilon(u, s)] \right| \geq c\varepsilon^{3-2\alpha+\vartheta}\right) \leq C\varepsilon^{2-\alpha-\vartheta}.$$

The estimate of $\Delta\mathcal{U}_\varepsilon$ follows by Gronwall's inequality from this result and equation (C.27)

$$\mathbb{P}\left(\sup_{0 \leq z \leq L, 0 \leq \Delta s \leq \varepsilon^\vartheta S} |\Delta\mathcal{U}_\varepsilon(z, s, \Delta s)| \geq c\varepsilon^{3-2\alpha+\vartheta}\right) \leq C\varepsilon^{2-\alpha-\vartheta}. \quad (\text{C.28})$$

Now we can bound $\mathcal{U}_\varepsilon(z, s)$ for all $s \in (s_{j-1}, s_j)$, with $s_j = j\varepsilon^\vartheta S$, using the triangle inequality

$$|\mathcal{U}_\varepsilon(z, s)| \leq |\mathcal{U}_\varepsilon(z, s_{j-1})| + |\Delta\mathcal{U}_\varepsilon(z, s_{j-1}, s - s_{j-1})|.$$

We obtain from (C.26), (C.28) and the assumption $\alpha \leq 1$ that

$$\sup_{z \in [0, L], s \in [s_{j-1}, s_j]} |\mathcal{U}_\varepsilon(z, s)| 1_{E_j} < c\varepsilon^\vartheta.$$

Here E_j is an event with probability $\mathbb{P}(E_j) \geq 1 - O(\varepsilon^{2-\alpha-\vartheta})$. There are $\varepsilon^{-\vartheta}$ such events and on their intersection $E = \cap_j E_j$, satisfying $\mathbb{P}(E) \geq 1 - O(\varepsilon^{2-\alpha-2\vartheta})$,

$$\sup_{z \in [0, L], s \in [0, S]} |\mathcal{U}_\varepsilon(z, s)| 1_E < c\varepsilon^\vartheta.$$

This is the statement (3.14).

C.4. Regime 2. The case $\beta \leq 1$, $\alpha = 2$, where

$$\mathcal{W}_\varepsilon(\tau, s) = \frac{1}{\varepsilon} \int_0^\tau du \mu \left(\frac{u}{\varepsilon^2} + s + \mathcal{W}_\varepsilon(u, s), \frac{c_o u}{\varepsilon^\beta} \right), \quad (\text{C.29})$$

can be handled with the change of variables

$$\zeta = u + \varepsilon^2 s + \varepsilon^2 \mathcal{W}_\varepsilon(u, s), \quad (\text{C.30})$$

so that

$$d\zeta = \left[1 + \varepsilon \mu \left(\frac{u}{\varepsilon^2} + s + \mathcal{W}_\varepsilon(u, s), \frac{c_o u}{\varepsilon^\beta} \right) \right] du. \quad (\text{C.31})$$

Since μ is bounded by assumption, we know from the implicit function theorem that we can invert (C.30) to get a unique function $u = u(\zeta)$, and with, $u = \zeta + O(\varepsilon^2)$. Substituting (C.30) in (C.29), we obtain that

$$\mathcal{W}_\varepsilon(\tau, s) \approx \frac{1}{\varepsilon} \int_0^\tau d\zeta \frac{\mu \left(\frac{\zeta}{\varepsilon^2}, \frac{c_o \zeta}{\varepsilon^\beta} - \varepsilon^{2-\beta} c_o (s + \mathcal{W}_\varepsilon(\zeta, s)) \right)}{1 + \varepsilon \mu \left(\frac{\zeta}{\varepsilon^2}, \frac{c_o \zeta}{\varepsilon^\beta} - \varepsilon^{2-\beta} c_o (s + \mathcal{W}_\varepsilon(\zeta, s)) \right)},$$

and expanding the right hand side in ε ,

$$\mathcal{W}_\varepsilon(\tau, s) \approx \frac{1}{\varepsilon} \int_0^\tau d\zeta \mu \left(\frac{\zeta}{\varepsilon^2}, \frac{c_o \zeta}{\varepsilon^\beta} - \varepsilon^{2-\beta} c_o (s + \mathcal{W}_\varepsilon(\zeta, s)) \right) - \int_0^\tau d\zeta \mu^2 \left(\frac{\zeta}{\varepsilon^2}, \frac{c_o \zeta}{\varepsilon^\beta} \right).$$

Here we used that μ and μ_z are bounded, and the approximate sign stands for equal up to absolute errors that tend to zero as $\varepsilon \rightarrow 0$. Now we can use the same analysis as in sections C.1-C.3 to show that we can write $\mathcal{W}_\varepsilon(\tau, s)$ as stated in Lemma 3.1.

Appendix D. Proof of Lemma 4.1. We wish to show that

$$\begin{aligned} \mathcal{T}^\varepsilon(\tau; s, y) = \int_0^\tau du \left[\mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \left(\frac{s+y}{2} + \mathcal{W}_\varepsilon(u, s) \right), \frac{c_o u}{\varepsilon^2} + \frac{c_o (s-y)}{2} \right) \times \right. \\ \left. \mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(u, s)), \frac{c_o u}{\varepsilon^2} \right) + \partial_z^2 \Phi \left(0, \frac{c_o (s-y)}{2} \right) \right] \quad (\text{D.1}) \end{aligned}$$

converges to 0 as $\varepsilon \rightarrow 0$, uniformly in $\tau \leq T$ and $0 \leq y \leq s \in [0, S]$. We write (D.1) as the sum $\mathcal{T}^\varepsilon = \mathcal{T}_1^\varepsilon + \mathcal{T}_2^\varepsilon$, where the first term is given by

$$\begin{aligned} \mathcal{T}_1^\varepsilon(\tau) = \int_0^\tau du \left[\mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \left(\frac{s+y}{2} + \mathcal{W}_\varepsilon(u, s) \right), \frac{c_o u}{\varepsilon^2} + \frac{c_o (s-y)}{2} \right) \times \right. \\ \left. \mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(u, s)), \frac{c_o u}{\varepsilon^2} \right) - \mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(u - \varepsilon^p, s)), \frac{c_o u}{\varepsilon^2} \right) \times \right. \\ \left. \mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \left(\frac{s+y}{2} + \mathcal{W}_\varepsilon(u - \varepsilon^p, s) \right), \frac{c_o u}{\varepsilon^2} + \frac{c_o (s-y)}{2} \right) \right], \end{aligned}$$

and the second term is

$$\begin{aligned} \mathcal{T}_2^\varepsilon(\tau) = & \int_0^\tau du \left[\mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \left(\frac{s+y}{2} + \mathcal{W}_\varepsilon(u - \varepsilon^p, s) \right), \frac{c_o u}{\varepsilon^2} + \frac{c_o(s-y)}{2} \right) \times \right. \\ & \left. \mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(u - \varepsilon^p, s)), \frac{c_o u}{\varepsilon^\beta} \right) + \partial_z^2 \Phi \left(0, \frac{c_o(s-y)}{2} \right) \right], \end{aligned}$$

with p as in (C.3). We suppressed s and y in the arguments to simplify notation. The mean value theorem and the definition of \mathcal{W}_ε allows us to bound $\mathcal{T}_1^\varepsilon$ as

$$|\mathcal{T}_1^\varepsilon| \leq C\varepsilon^{p+1-\alpha}, \quad (\text{D.2})$$

with constant C that is independent of ε , τ , s and y .

To estimate $\mathcal{T}_2^\varepsilon$ we use an argument that is similar to that used in Appendix C.2. We begin with

$$\mathbb{E} \left[\mathcal{T}_2^\varepsilon(m\varepsilon) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] = \sum_{j=1}^3 J_j, \quad (\text{D.3})$$

which we write as the sum of three terms. The first satisfies

$$|J_1| = \left| \mathbb{E} \left[\mathcal{T}_2^\varepsilon((m-1)\varepsilon + \varepsilon^2) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right| \leq C\varepsilon^2,$$

because μ_z is bounded. The second term is

$$J_2 = \mathbb{E} \left[\mathcal{T}_2^\varepsilon((m-1)\varepsilon + \varepsilon^p) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon + \varepsilon^2) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right],$$

and we can bound it using the mixing assumption (3.11). Since $\mathcal{W}_\varepsilon(u - \varepsilon^p, s)$ is $\mathcal{F}_{\frac{m-1}{\varepsilon}}$ measurable for $u \leq (m-1)\varepsilon + \varepsilon^p$, we obtain from (3.11) and the definition of $\mathcal{T}_2^\varepsilon$ that

$$\begin{aligned} \left| J_2 - \int_{(m-1)\varepsilon + \varepsilon^2}^{(m-1)\varepsilon + \varepsilon^p} dy \left[\partial_z^2 \Phi \left(0, \frac{c_o(s-y)}{2} \right) - \partial_z^2 \Phi \left(\frac{\varepsilon^{2-\alpha}(y-s)}{2}, \frac{c_o(s-y)}{2} \right) \right] \right| \leq \\ C \int_{(m-1)\varepsilon + \varepsilon^2}^{(m-1)\varepsilon + \varepsilon^p} \left(\frac{u}{\varepsilon^2} - \frac{m-1}{\varepsilon} \right)^{-d} = \frac{C\varepsilon^2}{d-1} \left(1 - \varepsilon^{(d-1)(2-p)} \right). \end{aligned}$$

The integrand in the left hand side is $O(\varepsilon^{2-\alpha})$, and since $p+2-\alpha > 2$ we get $|J_2| \leq C\varepsilon^2$. In the last term

$$J_3 = \mathbb{E} \left[\mathcal{T}_2^\varepsilon(m\varepsilon) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon + \varepsilon^p) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right]$$

we have $u - \varepsilon^p \geq (m-1)\varepsilon$ and use the tower property of the expectation to write

$$\begin{aligned} J_3 = & \mathbb{E} \left[\int_{(m-1)\varepsilon + \varepsilon^p}^{m\varepsilon} du \left\{ \mathbb{E} \left[\mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} \left(\frac{s+y}{2} + \mathcal{W}_\varepsilon(u - \varepsilon^p, s) \right), \frac{c_o u}{\varepsilon^2} + \frac{c_o(s-y)}{2} \right) \times \right. \right. \right. \\ & \left. \left. \mu_z \left(\frac{u}{\varepsilon^\alpha} + \varepsilon^{2-\alpha} (s + \mathcal{W}_\varepsilon(u - \varepsilon^p, s)), \frac{c_o u}{\varepsilon^2} \right) \middle| \mathcal{F}_{\frac{u-\varepsilon^p}{\varepsilon^2}} \right] + \partial_z^2 \Phi \left(0, \frac{c_o(s-y)}{2} \right) \right\} \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right], \end{aligned}$$

The inner expectation equals

$$-\partial_z^2 \Phi \left(\frac{\varepsilon^{2-\alpha}(y-s)}{2}, \frac{c_o(s-y)}{2} \right) + O(\varepsilon^{(2-p)d}) = -\partial_z^2 \Phi \left(0, \frac{c_o(y-s)}{2} \right) + O(\varepsilon^{2-\alpha}),$$

because $(2-p)d > 2 - \alpha$. Thus, $|J_3| \leq C\varepsilon^2$, and from (D.3) we find

$$\left| \mathbb{E} \left[\mathcal{T}_2^\varepsilon(m\varepsilon) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right| \leq C\varepsilon^2. \quad (\text{D.4})$$

Now let

$$Y_m = \mathcal{T}_2^\varepsilon(m\varepsilon) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon) - \mathbb{E} \left[\mathcal{T}_2^\varepsilon(m\varepsilon) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \quad (\text{D.5})$$

and define the martingale $X_m = \sum_{j=1}^m Y_j$. We have

$$\left| \mathcal{T}_2^\varepsilon(m\varepsilon) \right| \leq |X_m| + \sum_{j=1}^m \left| \mathbb{E} \left[\mathcal{T}_2^\varepsilon(m\varepsilon) - \mathcal{T}_2^\varepsilon((m-1)\varepsilon) \middle| \mathcal{F}_{\frac{m-1}{\varepsilon}} \right] \right| \leq |X_m| + Cm\varepsilon^2,$$

with the second bound following from (D.4). We also get from the definition of $\mathcal{T}_2^\varepsilon$ and the boundedness of μ_z that

$$\left| \mathcal{T}_2^\varepsilon(\tau) - \mathcal{T}_2^\varepsilon(m\varepsilon) \right| \leq C\varepsilon, \quad \forall \tau \in (m\varepsilon, (m+1)\varepsilon),$$

and therefore

$$\sup_{\tau \in [0, T]} |\mathcal{T}_2^\varepsilon(\tau)| \leq \sup_{m \leq T/\varepsilon} |X_m| + C\varepsilon. \quad (\text{D.6})$$

It remains to bound X_m , using the Submartingale Inequality

$$\mathbb{P} \left(\sup_{m \leq T/\varepsilon} |X_m| \geq c\varepsilon^{\frac{2-\theta}{2}} \right) \leq \frac{\varepsilon^{\theta-2}}{c^2} \mathbb{E} \left[X_{T/\varepsilon}^2 \right], \quad (\text{D.7})$$

for θ as in Lemma 3.1, where we suppose for simplicity that T/ε is integer. We have

$$\mathbb{E} \left[X_{T/\varepsilon}^2 \right] = \sum_{j=1}^{T/\varepsilon} \mathbb{E} \left[Y_j^2 \right] \leq 4 \sum_{j=1}^{T/\varepsilon} \mathbb{E} \left[|\mathcal{T}_2^\varepsilon(j\varepsilon) - \mathcal{T}_2^\varepsilon((j-1)\varepsilon)|^2 \right], \quad (\text{D.8})$$

where we used the tower property of the expectation and Jensen's inequality. To calculate the last expectation we denote in short by $n_\varepsilon(u)$ the integrand in the definition of $\mathcal{T}_2^\varepsilon$. Again, we suppress s and y for simplicity. We have

$$\begin{aligned} \mathbb{E} \left[\iint_{(m-1)\varepsilon}^{m\varepsilon} du dv n_\varepsilon(u) n_\varepsilon(v) \right] &= 2\mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} du n_\varepsilon(u) \int_u^{m\varepsilon} dv n_\varepsilon(v) \right] \\ &= 2\mathbb{E} \left[\int_{(m-1)\varepsilon}^{m\varepsilon} du n_\varepsilon(u) \mathbb{E} \left[\int_u^{m\varepsilon} dv n_\varepsilon(v) \middle| \mathcal{F}_{\frac{u}{\varepsilon^2}} \right] \right], \end{aligned}$$

with the conditional expectation bounded as in (D.4). Thus,

$$\mathbb{E} \left[|\mathcal{T}_2^\varepsilon(j\varepsilon) - \mathcal{T}_2^\varepsilon((j-1)\varepsilon)|^2 \right] \leq C\varepsilon^3,$$

and substituting in (D.7)-(D.8) we get the probabilistic bound on X_m . The estimate (D.6) becomes

$$\mathbb{P} \left(\sup_{\tau \in [0, T]} |\mathcal{T}_2^\varepsilon(\tau)| \geq c\varepsilon^{\frac{2-\theta}{2}} \right) \leq CT\varepsilon^\theta. \quad (\text{D.9})$$

Results (D.2) and (D.9) give that $\mathcal{T}^\varepsilon(\tau; s, y)$ converges in probability to 0, uniformly in τ and pointwise in s and y . More explicitly, assumption (C.3) gives $p + 1 - \alpha > 1 > (2 - \theta)/2$, so $\mathcal{T}_2^\varepsilon$ dominates $\mathcal{T}_1^\varepsilon$ and

$$\mathbb{P}\left(\sup_{\tau \in [0, T]} |\mathcal{T}^\varepsilon(\tau; s, y)| \geq c\varepsilon^{\frac{2-\theta}{2}}\right) \leq CT\varepsilon^\theta, \quad (\text{D.10})$$

for all s and y satisfying $0 \leq y \leq s \leq S$. Next we obtain from definition (D.1) and the assumptions on μ and Φ that

$$\sup_{\tau \in [0, T], 0 \leq y \leq s \leq S} |\partial_y \mathcal{T}^\varepsilon(\tau; s, y)| \leq C. \quad (\text{D.11})$$

The derivative $\partial_s \mathcal{T}^\varepsilon$ involves $\partial_s \mathcal{W}_\varepsilon$, for which we have the result in Lemma 3.1, so

$$\mathbb{P}\left(\sup_{\tau \in [0, T], 0 \leq y \leq s \leq S} |\partial_s \mathcal{T}^\varepsilon(\tau; s, y)| \geq c\right) \leq C\varepsilon^\theta. \quad (\text{D.12})$$

We discretize the interval $[0, S]$ in uniform steps ε^r with r to be specified below, and obtain

$$\mathbb{P}\left(\sup_{j \leq m \leq S/\varepsilon^r, \tau \in [0, T]} |\mathcal{T}^\varepsilon(\tau; m\varepsilon^r, j\varepsilon^r)| \geq c\varepsilon^{\frac{2-\theta}{2}}\right) \leq CT\varepsilon^{\theta-2r}. \quad (\text{D.13})$$

The factor ε^{-2r} in the bound is because we have ε^{2r} terms enumerated by (m, j) . In view of (D.11) and (D.12) we then find

$$\mathbb{P}\left(\sup_{y \leq s \leq S, \tau \in [0, T]} |\mathcal{T}^\varepsilon(\tau; s, y)| \geq c(\varepsilon^{\frac{2-\theta}{2}} + \varepsilon^r)\right) \leq CT\varepsilon^{\theta-2r}. \quad (\text{D.14})$$

Moreover, letting $T \mapsto T\varepsilon^q$, and absorbing the new T in the constant C , we get

$$\mathbb{P}\left(\sup_{y \leq s \leq S, \tau \in [0, T\varepsilon^q]} |\mathcal{T}^\varepsilon(\tau; s, y)| \geq c(\varepsilon^{\frac{2-\theta}{2}} + \varepsilon^r)\right) \leq C\varepsilon^{\theta-2r+q}. \quad (\text{D.15})$$

Finally, let $r = (\theta - \vartheta)/2$, for some ϑ satisfying $\max[0, 2(\theta - 1)] < \vartheta < \theta$, so that

$$r = \frac{\theta - \vartheta}{2} < \frac{2 - \theta}{2},$$

and therefore

$$\mathbb{P}\left(\sup_{y \leq s \leq S, \tau \in [0, T\varepsilon^q]} |\mathcal{T}^\varepsilon(\tau; s, y)| \geq c\varepsilon^{\frac{\theta-\vartheta}{2}}\right) \leq C\varepsilon^{\vartheta+q}. \quad (\text{D.16})$$

For φ a smooth function this then gives Lemma 4.1. NB any starting point

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